Math is Cool High School Championships – October 2009
9-12 Individual Test Solutions

1	9	48=3+5x, x=9
2	196π [cm^2]	$\pi r^2 = \pi 14^2 = 196\pi$
3	1458 [bunnies]	The total number of bunnies triples every 2 months. 2x3^6=1458
4	247 [pages]	Work backwards, 4470-1000=3470 divided by 500 gives 6.94 per book. Subtract 2.00 gives 4.94 / .02 = 247.
5	$4\sqrt{13}$ [cm]	$c^2 = a^2 + b^2$ $64 + 144 = 16(4+9) \quad c = 4\sqrt{13}$
6	16	The largest chord of a circle is the diameter, which is twice the length of the radius. 16.
7	74 [tickets]	175-94+1 minus 8 tickets ending in 0.
8	25	6 and one-fourth percent is 1/16. 1/16 of 400 is 25.
9	12 [edges]	An octahedron is composed of two square pyramids connect at their bases.
10	True	The perpendicular bisector will go through all the points equidistant from the ends of the chord – one of which is the center.
11	19/5	Square b first; order of operations.
12	12	There are three "different" placements for 'X'. $\begin{bmatrix} X & 1 & 2 \\ - & 3 & 4 \\ - & - & 5 \end{bmatrix}\begin{bmatrix} 6 & X & - \\ 7 & 8 & - \\ 9 & 10 & - \end{bmatrix}\begin{bmatrix} 11 & 12 & - \\ - & X & - \\ - & - & - \end{bmatrix}$
13	7	The successive powers end in 7, 9, 3, 1, 7, … Since 2009 has remainder 1 when divided by 4, 7^2009 will end in 7.
14	84	(23-16)12 = 7x12 = 84
15	1/221	$\frac{4}{52}\times\frac{3}{51} = \frac{1}{13}\times\frac{1}{17} = \frac{1}{221}$
16	385	The formula is n(n+1)(2n+1)/6= 10(11)(21)/6=385.
17	9	Let $x = 2\sqrt{3+2\sqrt{3+\ldots}}$, thus: $x = 2\sqrt{3+x}$ $x^2 = 4(3+x)$ $x^2 - 4x - 12 = 0$ $(x-6)(x+2) = 0$ $x = 6$ $x + 3 = 9$

18	x + 3y = 5	The slope of the original line is 3 so the perpendicular line is -1/3. Fit y=-x/3+b to the point (-1,2), b=5/3. Then multiply by 3 and simplify to x + 3y=5. Shortcut: interchange the coeff. of x and y and change one sign, then fit the constant to the point.
19	75 [mph]	The average speed is given by $ave = \frac{2ab}{a+b}$. So $60 = \frac{2(50)b}{50+b}$ and b=75.
20	144	24=4*6, 4 is already square so 24*6=144 is the smallest square.
21	4π or 4 pi	Complete the squares and divide by the coefficients of the square terms gives: $\frac{(x-1)^2}{2}+\frac{(y+2)^2}{8}=1$. So ab=4 and the area is 4pi.
22	1	Let the answer be x, so x=2/(1+x) or x^2+x-2=0. Therefore x=-2 or 1 – choose the positive solution.
23	320 [ways]	5 choices for the first stripe and then 4 for each of the other 3. 5x4x4x4=320
24	35	100%=2.5(40%), so 2.5(14)=35 students total.
25	15	The sum of 3 sides must be greater than the 4th. So, the smallest it can be is 2 and the largest is 16.
26	15π	$\frac{25\min}{60\min}\pi(6in)^2 = 15\pi\, sq.in.$
27	12,960	There are 30 possibilities for the first number, 24 for the second and 18 for the third. (30)(24)(18)=12960.
28	6	This means the distance between 2m and 5 is less than 6 so 2m must be between -1 and 11. m is 0,1,2,3,4 or 5.
29	18 [chickens]	18 chickens can then lay 10 eggs in two days and 20 in four days.
30	13/7	The dot product (x+1)2+5(x-3)=0 solving yields x=13/7.
31	2/3	Change all bases to 2 and divide by the RHS to get: $2^{3x^2+7x-6}=2^0$. (3x-2)(x+3)=0
32	60	The determinant of the product is the product of determinants, so the answer is: [(4(3)-2(1)][2(3)-6(0)]=60
33	7/125	There are a total of 125 possibilities (5x5x5) and six ways to get six (1,2,3 in any order) and 2,2,2.
34	$(-\infty,-2)\cup\left(\frac{3}{2},2\right)$	Subtract the RHS and simplify, factor to get $\frac{(2x-3)(z+2)}{2(z-2)}<0$. For the product to be < 0, one or all three must be < 0.
35	(1,1,0)	Divide numerator and denominator by (x+3) which causes a hole in the graph at x=-3. The resultant function (x-2)/(2x+1) has a vertical asym. at x=-1/2 and horizontal asym at $\frac{1}{2}$.

36	1	The terms are: $3^{1/2}, 3^{1/3}, 3^{1/6}$. The common ratio is $3^{-1/6}$ which makes the next term $3^0 = 1$.
37	175/396	This is the hypergeometric distribution. There are 5C2=10 ways of picking the blue and 7C3=35 ways for the red and 12C5 ways overall.
38	20 [ways]	Take 2 chairs away and let the 3 students sit in any chairs they like. This can be done in 6C3=20 ways. Replace the two chairs between the students.
39	2	Start with $\sqrt{49+12\sqrt{10}} = \sqrt{x} + \sqrt{y}$ $49 + 2\sqrt{360} = x + y + 2\sqrt{xy}$ so $x = 9, y = 40$ $\sqrt{4+\sqrt{9}+\sqrt{40}} = \sqrt{a} + \sqrt{b}$ $7 + 2\sqrt{10} = x + b + 2\sqrt{ab}$ $a = 2, b = 5$
40	6,8 [in any order]	c/a=4(12)=48; -b/a=4+10=14. The sum of the roots is 14 and the product is 48. The roots are then 6 and 8.

Math is Cool High School Championships – October 2009
9-10 Individual Multiple Choice Test Solutions

1	A	$\frac{4}{5}x = 2(x-3)$ $= 2x - 6$ $6 = x\left(2 - \frac{4}{5}\right)$ $6 = \left(\frac{6}{5}\right)x$ $x = 5$
2	C	Rewrite each term as the difference of two fractions. [1/1 – 1/2]+[1/2 – 1/3] + … + [1/19 – 1/20] = 1 – 1/20 = 19/20.
3	B	There are 15 ways of obtaining a sum of 7 with a roll of 3 dice, out of 216. (5,1,1), (4,2,1), (4,1,2), (3,3,1), (3,1,3), (3,2,2), (2,4,1), (2,3,2), (2,2,3), (2,1,4), (1,5,1), (1,4,2), (1,3,3), (1,4,2), (1,5,1).
4	A	Each piece of the sum must be zero giving two equations in x and y. Solving yields: x=2,y=-9
5	D	Prime factorize 780: $780 = 2^2 3^1 5^1 13^1$. The expansion of the following expression yields the sum of all factors of 780: $(2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1)(13^0 + 13^1)$ $S = (1+2+4)(1+3)(1+5)(1+14)$ $= (7)(4)(6)(14)$ $= 2352$
6	E	In the plane determined by the three points, there is a single point (the circumcenter). In space, the points form the line perpendicular to the plane going through the point.
7	B	There are 8C3=56 ways to choose three squares and 5 ways to get three in a row. 2 vertical, 2 horizontal and 1 diagonal.
8	D	The number of arrangements of the letters in the word MANHATTAN is equivalent to: $\frac{9!}{3!2!2!} = 15120$
9	B	There are six ways to reorder the outfits; only TFW and FWT have all the outfits in the wrong order.

Math is Cool High School Championships – October 2009
11-12 Individual Multiple Choice Test Solutions

1	A	$\frac{4}{5}x = 2(x-3)$ $= 2x - 6$ $6 = x\left(2 - \frac{4}{5}\right)$ $6 = \left(\frac{6}{5}\right)x$ $x = 5$
2	C	Rewrite each term as the difference of two fractions. [1/1 – 1/2]+[1/2 – 1/3] + … + [1/19 – 1/20] = 1 – 1/20 = 19/20.
3	B	There are 15 ways of obtaining a sum of 7 with a roll of 3 dice, out of 216. (5,1,1), (4,2,1), (4,1,2), (3,3,1), (3,1,3), (3,2,2), (2,4,1), (2,3,2), (2,2,3), (2,1,4), (1,5,1), (1,4,2), (1,3,3), (1,4,2), (1,5,1).
4	C	The function sin(2x) has period π and cos(3x) has period $2/3\,\pi$; the least common multiple is 2π.
5	D	Prime factorize 780: $780 = 2^2 3^1 5^1 13^1$. The expansion of the following expression yields the sum of all factors of 780: $(2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1)(13^0 + 13^1)$ $S = (1+2+4)(1+3)(1+5)(1+14)$ $= (7)(4)(6)(14)$ $= 2352$
6	E	In the plane determined by the three points, there is a single point (the circumcenter). In space, the points form the line perpendicular to the plane going through the point.
7	B	Use the identity: $\frac{1}{\log_a(b)} = \log_b(a)$ on each term. Since the base is now the same, we can combine to get $\log_{20!}(20!) = 1$
8	D	$d = 8\begin{vmatrix} 4 & 1 \\ 6 & 9 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 6 & 9 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ Expansion by minors, repeatedly: $= 8(36-6) + 1(3-8)$ $= 240 - 5 = 235$
9	B	There are six ways to reorder the outfits; only TFW and FWT have all the outfits in the wrong order.

Math is Cool High School Championships – October 2009
9-10 Team Test Solutions

1	145 [degrees]	$h = 2(30) + \frac{5}{6}(30) = 85$ $m = \frac{5}{6}(360) = 300$ $360 + 85 - 300 = 145\,\text{deg}$
2	95 [hippos]	$h = 3i - 4;\ \ i = \frac{1}{2}(h - 29);\ h = 95$
3	5/2	The product of the roots is –(-5)/2.
4	88	$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$
5	23 [cents]	5(7)-5-7=35-12=23. 24=2(5)+2(7), 25=5(5), 26=1(5)+3(7), …
6	8 [cm]	Draw radii from the centers to the point of tangency and a line parallel to the tangent through the center of the smaller circle. Connecting the centers of the circles make a right triangle with sides, (13-7)=6, x and hypotenuse 10. x=8.
7	TAEHT	There are 12 permutations starting with A (4!/2!), same for E and H. The 37th one with be the first one beginning with T; TAEHT.
8	$1332101_{[6]}$	One can repeatedly divide by 6, with the dividend in base 7. $6\overline{)1r3}\ r1$ $6\overline{)12r3}$ $6\overline{)111r2}$ $6\overline{)1001r1}$ $6\overline{)6010r0}$ $6\overline{)51060r1}$ $6\overline{)426511}$
9	9	We have 3(62) + 7(2) = 200. Since 3 and 7 are relatively prime, the next pair will be at 3(55) + 7(5). a = 62, 55, 48, …, 6.
10	$m/3 - 2n$ or $\frac{m-6n}{3}$	The average of the elements is the same as the 5th element = m/9, so n – m/9 = 2d, where d is the common difference. Solve for d and subtract 6d from n to get the 1st element. $n - \left(3n - \frac{m}{3}\right) = \frac{m}{3} - 2n = \frac{m-6n}{3}$

Math is Cool High School Championships – October 2009
11-12 Team Test Solutions

1	145 [degrees]	$h = 2(30) + \frac{5}{6}(30) = 85$ $m = \frac{5}{6}(360) = 300$ $360 + 85 - 300 = 145\,deg$
2	95 [hippos]	$h = 3i - 4;\ \ i = \frac{1}{2}(h - 29);\ h = 95$
3	2520	We need f to be 1-1 meaning no two elements of A map to the same element in B. So there are 7 choices for the first element of A, 6 for the second, etc. 7x6x5x4x3=2520.
4	88	$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$
5	25	$9^{\log_3 5} = 3^{2\log_3 5} = 3^{\log_3 25} = 25$
6	8 [cm]	Draw radii from the centers to the point of tangency and a line parallel to the tangent through the center of the smaller circle. Connecting the centers of the circles make a right triangle with sides, (13-7)=6, x and hypotenuse 10. x=8.
7	TAEHT	There are 12 permutations starting with A (4!/2!), same for E and H. The 37th one with be the first one beginning with T; TAEHT.
8	$1332101_{[6]}$	One can repeatedly divide by 6, with the dividend in base 7. $\begin{array}{rr} & r1 \\ 6 & \overline{)1r3} \\ 6 & \overline{)12r3} \\ 6 & \overline{)111r2} \\ 6 & \overline{)1001r1} \\ 6 & \overline{)6010r0} \\ 6 & \overline{)51060r1} \\ 6 & \overline{)426511} \end{array}$
9	250	Let a and b be the two diagonals and C be one of the angles. By the law of cosines used twice: $a^2 = 5^2 + 10^2 - 2(5)(10)\cos(C)$ $b^2 = 5^2 + 10^2 - 2(5)(10)\cos(180 - C)$ Cosines of C is the negative of the cosine of 180-C, so that adding the two equations gives: $a^2 + b^2 = 250$
10	$m/3 - 2n$ or $\frac{m - 6n}{3}$	The average of the elements is the same as the 5th element = m/9, so n – m/9 = 2d, where d is the common difference. Solve for d and subtract 6d from n to get the 1st element. $n - \left(3n - \frac{m}{3}\right) = \frac{m}{3} - 2n = \frac{m - 6n}{3}$

Math is Cool High School Championships – October 2009

9-12 Pressure Round Test Solutions

1	19/5	$g(7) = g(f(2)) = \frac{2(8)+2+1}{4+1} = \frac{19}{5}$
2	56	Using the difference between two squares: [91319^2-5^2]-[91319^2-9^2]=81-25=56
3	9	Just multiply the factors and look at the coefficients of x^2 and the constant term.
4	28 [games]	One can of course use algebra: (6+x)/(20+x)=1/2. Or we can notice we have lost 14 games and must win at least 8 more to be at 50% wins.
5	40/3 [min]	There are two equations for the distance between the houses: D=8T+8S; D=10T+5S. Multiply the first equation by 5 and the second by 8 and subtract. 3D=40T or D=(40/3)T.

Math is Cool High School Championships – October 2009
9-12 Mental Math Test Solutions

1.1	4/3	The slope of the line is -3/4, the negative reciprocal is 4/3.
1.2	5	Solve in reverse (7 + 8) / 3 = 5.
1.3	20	(10^10)^3 / (10^5)^2
1.4	3/11	There are 11 ways that one can get at least one 6, and 3 of those have a sum of at least 11.
2.1	Yes or True	
2.2	15	(106-8)/7 + 1 = 98/7 + 1 = 14+1=15.
2.3	27	There are 2 outer edges on each face. Only the inner 3x3x3 are not painted. 27
2.4	3	(7,23), (11,19), (13,17)
3.1	44	The "plus 17" cancels and the 4 (6^2) – 4 (5^2) = 4 (6-5)(6+5)=44.
3.2	35/37	Multiply num and den by 6 before combining. (36-1)/(36+1)=35/37.
3.3	8	The hypotenuse is length 2root2
3.4	.58	sqrt(1/3)=sqrt(3)/3 .= 1.732/3 .= .58.
4.1	18	Rather than computing the side of the square, the area is ½ the product of the diagonals which are the same. (1/2)(6)(6)=18.
4.2	$1\frac{3}{4}$ or One and three fourths	$\sqrt{3\frac{1}{16}} = \sqrt{\frac{49}{16}} = \frac{7}{4} = 1\frac{3}{4}$
4.3	63	(36/4) sets of 7. 9x7=63.
4.4	1332	When the 6 numbers are written in a column, each of the three digits will appear twice in each place. 2(1+2+3)=12. A little carrying and the total is 1332.

Math is Cool High School Championships – October 2009
9-12 College Bowl A Solutions

1	25.47	28.3 – 2.83 = 25.47
2	Centroid	48=3+5x, x=9
3	18 [cm^2]	4x9/2=18
4	1/2	[2(2)-3]/2 = 1/2
5	2^3*3*5 or 2 cubed times 3 times 5	
6	3780	9 letters with 4 E's, 2 N's and 2 S's. 9!/[4!2!2!]=3780
7	8	[1+4+5+5+25]/5= 40/5=8
8	29	X+22=57-6, x=29.

1	17 and 1/2	$\frac{25}{6}\times\frac{21}{5}=\frac{35}{2}=17\frac{1}{2}$
2	15	$\frac{96}{4}-9=24-9=15$
3	15 pi [cm^2]	$\frac{150}{360}\pi(6^2)=\frac{5}{12}(36)\pi=15\pi$
4	2160	$\binom{6}{4}(-2)^4 3^2=2160$
5	$12422_{[5]}$	Repeatedly divide by 5, remainders are the digits of the number.
6	7	Number the seats, 1,2, …, 8. If 1,2 are painted, the third can 3,4,5,6,7. 123 is the same as 812. If 1,3 are painted the third can be 5 or 6. 7 possibilities.
7	27	$18\frac{1}{1-1/3}=18\frac{3}{2}=27$
8	90 [diagonals]	15(15-3)/2=90

1	64 [%]	$\frac{100}{150}(96)=\frac{2}{3}(96)=64$
2	2520	2^3 x 3^2 x 5 x 7 = 2520 9x8x7x5 is also a multiple of 10,6,4,3,2 and 1.
3	4 + 4 root 3 [cm^2]	Area of a equilateral triangle is $\frac{s^2\sqrt{3}}{4}$. Total surface area is: $2(2)+4\frac{2^2\sqrt{3}}{4}=4+4\sqrt{3}$
4	[$] 31,827	1.03x1.03x30,000 = 1.0609 x 30,000 = 31,827
5	12	$350=2\bullet5^2\bullet7$; (2)(3)(2) = 12
6	144	4! ways of ordering the CD's. Treating them as a unit, there are 3! ways of ordering them and the DVDs. 4!x3! = 144.
7	130	3, 4, 6, 10, 18, 34, 66, 130.
8	16	1+i squared is 2i, 2i to the fourth is 16.

Math is Cool High School Championships – October 2009
9-12 College Bowl B Solutions

1	30240 [min]	(60)(24)(21)=30240
2	13 [rattlesnakes]	38/2 = 19 ostriches. 32-19 = 13 snakes.
3	300pi [cm^2]	Slant height is found using Pythagoras and is 13. Lateral surface area is (12)(13)pi=156pi. Base is 144pi. Total=300pi.
4	27	u=k/v, k=uv=18(144). v=k/u=(18)(144)/96 = 27
5	240	gcd(30,48)=6. (30)(48)/6=(30)8 = 240.
6	317	(418+356) – (528-71) = 317.
7	16	1000/37=27+. Subtract 2 for 37,74. Subtract 9 more for 111, 222, etc. = 16.
8	1/3	There are 5C3 = 10 ways of getting 3 heads. Minus the all heads and all tails cases, there are 30 possibilities. 10/30 = 1/3.

2	106	Units digit must be 5 more than hundreds. 106 is first possibility.
3	42+6root29 [cm]	Factor out the 6. Perimeter = $6[2+5+\sqrt{29}] = 42+6\sqrt{29}$
4	-13	$f(3) = 3(3^2) - 5 \bullet 2^3 = 27 - 40 = -13$
5	900	9 choices for 1st digit, 10 for the 2nd and 3rd. The 4th must be the same as the 2nd and the 5th must be the 1st. 9(10)(10) = 900
6	1/4	(4C3)/16=4/16=1/4
7	1050	$5+10+\ldots+100 = 5(1+2+\ldots+20)$ $= 5\frac{20(21)}{2} = 5(210) = 1050$
8	16	849+1 = 850 = 17(50), so the remainder is 16.

1	12+3 root3	$\frac{39(4+\sqrt{3})}{16-3} = 12+3\sqrt{3}$

1	23	7+4+12 = 23
2	1800 [km]	6 (300) = 1800
3	16 [cm]	Two sides must add to more than the third. 9+16=25>24.
4	(2,-4)	Just pay attention to the x and y terms. (x-2)^2+(y+4)^2 = …. The center is (2,-4)
5	$1207_{[8]}$	Add as usual, remember to carry at 8 (instead of 10). $1207_{[8]}$.

6	48 [feet]	The total distance that the ball travels is the total upward distance plus the total downward distance. Therefore: $U+D=\frac{24}{1-\frac{1}{3}}+\frac{8}{1-\frac{1}{3}}$ $=36+12$ $=48$
7	195	$72=2^3\bullet 3^2$ (1+2+4+8)(1+3+9) = (15)(13) = 195

8	2 root 6 [cm]	Find area using Heron's formula. $s=(4+5+7)/2=8$ $A=\sqrt{8(1)(3)(4)}=4\sqrt{6}$ Area is also (1/2)bh=2h, so $h=2\sqrt{6}$