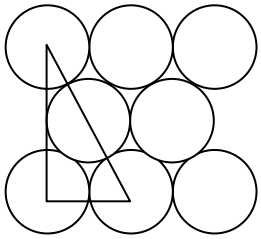
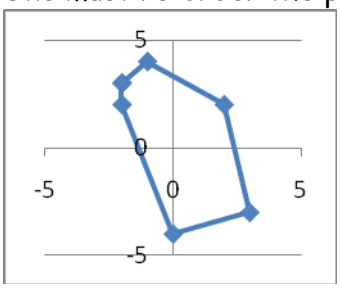
Math is Cool High School Championships – October 2010
9-12 Individual Test Solutions

1	56477	Just add
2	$\frac{4}{5}$	Cancel before you multiply to make it easier.
3	20	Sqrt((10- -2)^2+(-11-5)^2) = 20, It is a multiple of a 3-4-5 triple.
4	17	PEMDAS order of operations
5	10/7	Combine x's on one side and constants on the other, then divide.
6	5/2	Change to slope-intercept form by putting y on the other side of the equation.
7	539π	$V = \pi r^2 h$
8	-22, 16	Either x+3=19 or x+3 = -19.
9	13	Total is 91 divided by 7 numbers = 13.
10	$\frac{5x-9}{6}$ or $\frac{5x}{6}-\frac{3}{2}$	Common denominator is 6. $\frac{2x}{6}+\frac{3x-9}{6}=\frac{5x-9}{6}$
11	298	2(4x7 + 4x11 + 7x11) = 298
12	1560	40 ways of picking 1st place and 39 ways for 2nd place, order matters. 40(39) = 1560
13	[$]42	One and one-fifth times 35.
14	(x-6)(x+3)	Need two numbers that multiply to -18 and add to -3: -6 and 3.
15	2/3	(7+3)/(7+3+5) = 2/3
16	$\frac{25}{6}$	The sum of the roots will be: -(-25)/6
17	$\frac{105}{512}$	$\frac{10C6}{2^{10}}=\frac{210}{1024}=\frac{105}{512}$
18	6720 [ways]	8!/3! = 6720
19	2	All one has to do is substitute 1 for x to get a remainder of 2.
20	32°	The hour moves ½ degree every minute and starts at 120 degrees. The minute hand moves 6 degrees every minute and starts at 0 degrees. 128-96 = 32 degrees.
21	22	$lw = 28$ and $l^2 + w^2 = 65$. Since $(l+w)^2 = l^2 + w^2 + 2lw$ and $l + w = 11$ the perimeter is 22.
22	21 [lines]	Between any two points, one can draw a line: 7C2 = 21.

23	0	Even after the first matrix multiply, one can see the matrix will be 0 at (1,3).
24	$\frac{5}{13}$	Consider the first round: Matt wins, P(M)=1/6 Stacey wins: P(S)=(5/6)(1/3)=5/18. Bertha wins: P(B)=(5/6)(2/3)(1/2)=5/18. So the chance that someone wins is 13/18 and the portion of the time that Bertha wins is (5/18)/(13/18)=5/13. One can also do the infinite geometric series.
25	24 [cm]	The area of the annulus will be the area of the larger circle minus the area of the smaller circle. One can draw the right triangle from the center to the point of tangency to the end of the chord. One side is R, another is r and the third which is half the length of the chord is R^2-r^2 by Pythagoras.
26	$5\sqrt{5}$ [cm]	Use the height of the cylinder as one side and unroll it 5 times to get the other leg of a right triangle. The length of the line can be found using Pythagoras to be $\sqrt{10^2+5^2}=5\sqrt{5}$
27	343	If we expand out the expression and then set x=y=1, we get the sum of the coefficients. So the sum is $(2+5)^3=343$
28	60 [m]	Separate out the first 30 foot drop. The rest of the distance is twice (up and down) the infinite series $10+\frac{1}{3}10+\left(\frac{1}{3}\right)^2 10+\left(\frac{1}{3}\right)^3 10+\ldots=10\frac{1}{1-\frac{1}{3}}=15.$ Total is 30 + 2(15) = 60.
29	2	7!=5040, one divisibility rule for 11 is that 5040 has the same remainder as 50+40=90 and that is 2.
30	$\frac{8}{21}$	From the 4 guys, you need to choose 3 and from 6 girls you need to choose 3. There are 10 choose 6 ways overall to choose the committee. 4C3*6C3/10C6 = 8/21
31	8 [pieces]	Probably can be guessed.  The right triangle drawn will be a 30-60-90 triangle with hypotenuse equal to 4 and one leg 2. The height of the triangle is then $2\sqrt{3}$ which is less than 3.5 which makes the total height less than 5.5.

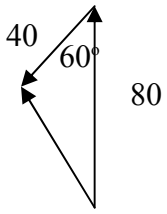
32	$2\sqrt{6}+4$ [inches]	If the 4 centers are connected, the resulting shape is a regular tetrahedron with side length $2\sqrt{6}$ The top vertex is directly above the centroid of the base which is 2/3 of the length of the median (altitude). A right triangle is drawn from a vertex on the base to the centroid of the base to the top vertex. The height of the tetrahedron is then: $h=\sqrt{\left(2\sqrt{6}\right)^2-\left(2\sqrt{6}\,\frac{\sqrt{3}}{2}\,\frac{2}{3}\right)^2}=\sqrt{24-8}=4$ Adding the top and bottom, we get $2\sqrt{6}+4$
33	6	Taking the first few powers: 5, 25, 125, 625, 3125, we see even power past 2 will be a 6.
34	$x+2$	When dividing by a quadratic polynomial, the remainder will be linear so p(x) can be written: $p(x)=(x^2-1)q(x)+Ax+B$. Note $p(1)=3=A+B$ $p(-1)=1=-A+B$ Solving for A and B gives a remainder of x+2.
35	36 [ways]	Line up the ten books in a row leaving a space between each two books. Of these 9 gaps between books, choose two. The books to the left of the first gap go to the first librarian, etc. So the number of ways of dividing the ten books is the same as the number of ways of choosing 2 gaps from 9. 9C2 = 36.
36	$\frac{49}{2}$ [units^2]	One must re-order the points to make a convex shape.  The "shoelace" method can be used to find the area.

	-2	3	
-3	-1	4	-8
8	2	2	-2
6	3	-3	-6
0	0	-4	-12
8	-2	2	0
-4	-2	3	-6
15			-34

Area = (15--34)/2 = 49/2

37	$\frac{37}{180}$	Use partial fractions to rewrite the sum as: $\sum_{x=1}^{\infty}\left(\frac{1}{x+3}-\frac{1}{x+6}\right)=\frac{1}{4}-\frac{1}{7}+\frac{1}{5}-\frac{1}{8}+\frac{1}{6}-\frac{1}{9}+\frac{1}{7}-\frac{1}{10}+\ldots$ $=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{37}{180}$
38	57 [lockers]	It is a little easier to count the ones that are open. All the locker numbers with 1 prime factor will be open; primes or powers of primes. We get 2,4,8,16,32,64, 3,9,27,81, 5,25, 7,49 and the other 21 primes for a total of 35. The others will be multiples of the product of 3 primes, 2*3*5=30,60,90; 2*3*7=42, 84 and 2*3*11=66, 2*3*13=78, 2*5*7=70, eight more for a total of 43 leaving 57 lockers still closed.
39	$\frac{4}{53}$	Let A be the first cup being bitter and B the second cup bitter. $p(A)=\frac{7}{8}, p(B\mid A)=\frac{7}{9}, p(B\mid A^c)=\frac{4}{9}$ $p(BA)=\frac{7}{8}\frac{7}{9}=\frac{49}{72}, p(BA^c)=\frac{4}{9}\frac{1}{8}=\frac{4}{72}$ $p(B)=\frac{49}{72}+\frac{4}{72}=\frac{53}{72}$ $p(A^c\mid B)=\frac{4}{72}/\frac{53}{72}=\frac{4}{53}$
40	478	We know 81a + 9b is divisible by 9, then so is 395 – c which makes c = 8. Dividing by 9, we get 9a + b = 43 so 43-b must also be divisible by 9 making b = 7. Solve for a=4 and the 3-digit number is 478.

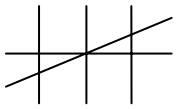
Math is Cool High School Championships – October 2010
9-10 Individual Multiple Choice Test Solutions

1	B	2+15-2 = 15
2	D	3+5+12 + 9+21 = 50.
3	A	5(5-3)/2=5
4	A	$\frac{12^2\sqrt{3}}{4} = 36\sqrt{3}$
5	C	$1152 = 2^7 3^2$, $8(3) = 24$
6	C	3 5 0 5 3 2 0 2 2 0 2 5 3 4
7	B	$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$
8	A	$11^{11} = (10+1)^{11}$ $= 10^{11} + \ldots + \binom{11}{3}10^3 + \binom{11}{2}10^2 + 11(10) + 1$ $= \ldots + ..5(1000) + 5500 + 110 + 1 = \ldots 0611$
9	D	6,5,1 – 6 ways 6,4,2 – 6 ways 6,3,3 – 3 ways 5,5,2 – 3 ways 5,4,3 – 6 ways 4,4,4 – 1 way Total of 25 ways over 6(6)(6)=216 possible rolls.
10	E $40\sqrt{3}$	 One can use trig or notice that a horizontal line drawn where the two arrows meet creates a 30-60-90 triangle, and then use Pythagoras on the lower triangle.

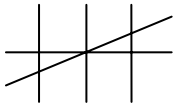
Math is Cool High School Championships – October 2010
11-12 Individual Multiple Choice Test Solutions

1	B	2+15-2 = 15
2	D	3+5+12 + 9+21 = 50.
3	A	5(5-3)/2=5
4	B	Law of cosines: $c^2 = 4^2 + 5^2 - 2(4)(5)\cos(60^\circ) = 21$
5	C	$1152 = 2^7 3^2$, $8(3) = 24$
6	C	3 5 0 5 3 2 0 2 2 0 2 5 3 4
7	B	The empty seat can be anywhere, going clockwise place the 4 couples in any one of 4! ways. Then, each couple can be switch in 2 ways. 4!(2^4)=384
8	A	$11^{11} = (10+1)^{11}$ $= 10^{11} + \ldots + \binom{11}{3}10^3 + \binom{11}{2}10^2 + 11(10) + 1$ $= \ldots + ..5(1000) + 5500 + 110 + 1 = \ldots 0611$
9	D	6,5,1 – 6 ways 6,4,2 – 6 ways 6,3,3 – 3 ways 5,5,2 – 3 ways 5,4,3 – 6 ways 4,4,4 – 1 way Total of 25 ways over 6(6)(6)=216 possible rolls.
10	E $\frac{49}{25}$	$\cos(\theta) = 3/5, \sin(\theta) = 4/5$ and $1 + \sin(2\theta) = 1 + 2\sin(\theta)\cos(\theta) = 1 + \frac{3}{5}\frac{4}{5} = \frac{49}{25}$

Math is Cool High School Championships – October 2010
9-10 Team Test Solutions

1	30	6!/4!
2	10	2, 3, and 5 are the prime factors, Sum=10.
3	$207_{[10]}$	$5*6^2+4*6+3=207$
4	-289	17*(41+(41+33*-3))=17*(-17)=-289
5	12	Three parallel lines, one perpendicular, and one at 40 degrees to the parallels. 
6	6	Medium: 3600=2^4*3^2*5^2 & 72=2^3*3^2, so there can be 3 or 4 2's, exactly 2 3's, and 0, 1, or 2 5's, for an answer of 2*3.
7	$(2,5)$	Complete the square $4(x^2-4x+4)-2(y^2-10y+25)=a$ $4(x-2)^2-2(y-5)^2=a$ Center is then (2,5).
8	330	n is of the form a(a-1), giving 2, 6, 12, 20, 30, 42, 56, 72, and 90. $\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}=k$ $\sqrt{n+k}=k$, use quadratic formula $1+\sqrt{1+4n}=2m$, solve for n $n=m(m-1)$
9	84	Ignore the shaded box. The total number of rectangles is 5C2 times 5C2 since there are 5 vertical and 5 horizontal lines and we choose 2 of each. (5C2)(5C2)=10(10)=100. Those including the shaded box must use the bottom and right lines. There are 4 ways each of picking the top and left lines. 100 – 4(4) = 84.
10	$\frac{3+2a}{2+a}$	$\log_{12}72=\frac{\log72}{\log12}=\frac{\log(8*9)}{\log(4*3)}$ $=\frac{3\log2+2\log3}{2\log2+\log3}=\frac{3+2a}{2+a}$

Math is Cool High School Championships – October 2010
11-12 Team Test Solutions

1	30	6!/4!
2	-5/12	cos x = -sqrt(1 – (5/13)^2) = -12/13 since 2nd quad. Sin/cos =-5/12
3	$207_{[10]}$	$5*6^2+4*6+3=207$
4	-289	17*(41+(41+33*-3))=17*(-17)=-289
5	12	Three parallel lines, one perpendicular, and one at 40 degrees to the parallels. 
6	8	$17^{17}=(18-1)^{17}=18^{17}-17(18^{16})+\ldots+17(18)-1$ All of the terms are divisible by 9 except the last; the -1 means the remainder will be 8.
7	$(2,5)$	Complete the square $4(x^2-4x+4)-2(y^2-10y+25)=a$ $4(x-2)^2-2(y-5)^2=a$ Center is then (2,5).
8	330	n is of the form a(a-1), giving 2, 6, 12, 20, 30, 42, 56, 72, and 90. $\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}=k$ $\sqrt{n+k}=k$, use quadratic formula $1+\sqrt{1+4n}=2m$, solve for n $n=m(m-1)$
9	84	Ignore the shaded box. The total number of rectangles is 5C2 times 5C2 since there are 5 vertical and 5 horizontal lines and we choose 2 of each. (5C2)(5C2)=10(10)=100. Those including the shaded box must use the bottom and right lines. There are 4 ways each of picking the top and left lines. 100 – 4(4) = 84.
10	$\frac{3a-a^3}{2}$	We know $(\sin x+\cos x)^2=\sin^2 x+\cos^2 x+2\sin x\cos x=a^2$ so $\sin x\cos x=\frac{a^2-1}{2}$. $\sin^3 x+\cos^3 x=(\sin x+\cos x)(\sin^2 x+\cos^2 x-\sin x\cos x)$ $=a\left(1-\frac{a^2-1}{2}\right)=\frac{3a-a^3}{2}$

Math is Cool High School Championships – October 2010
9-10 Pressure Round Test Solutions

1	4	983, 974, 965, 875
2	$\frac{1}{13}$	The first card can be anything. Of the 39 remaining cards of a different suit, 3 will make a pair. 3/39 = 1/13.
3	87	The congruence statement means that 257 and j differ by a multiple of 34. Take 257-5*34=87.
4	10 [oz]	One could guess at the answer that calculating what I would have each morning or set up an equation. If G is the amount of gold then: $G = \frac{G}{2} + 5$ since I would have the same amount two days in a row. The solution is 10.
5	6	$(3^g - 9)(3^g - 27) = 0$ g is then 3 and 2.

Math is Cool High School Championships – October 2010
11-12 Pressure Round Test Solutions

1	80640	The prime factors of 388080 are 2, 3, 5, 7 and 11. Half of the numbers less than 388080 are multiples of 2 and thus not relatively prime to 388080. The total number of relatively prime is then: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)388080$ $=\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{10}{11}\right)388080$ $=\frac{16}{77}388080=80640$
2	$\frac{1}{13}$	The first card can be anything. Of the 39 remaining cards of a different suit, 3 will make a pair. 3/39 = 1/13.
3	87	The congruence statement means that 257 and j differ by a multiple of 34. Take 257-5*34=87.
4	$\frac{16}{3}$[ft]	Solutions will be of the form: $t_n = a^n$ $t_0 = 4,\ t_1 = 6,\ 2t_{n+2} = t_n + t_{n+1}$ so $2a^2 = 1 + a,\ a = 1, -\frac{1}{2}$ $t_n = \left(-\frac{4}{3}\right)\left(-\frac{1}{2}\right)^n + \frac{16}{3} \to \frac{16}{3}$
5	6	$(3^g - 9)(3^g - 27) = 0$ g is then 3 and 2.

Math is Cool High School Championships – October 2010
9-12 Mental Math Test Solutions

9/10	11/12	Answer	Solution
1.1	1.1	11	4x+13=57, 4x=44, x=11
1.2	1.2	15	P=2l+2w, 2(4)+2(7/2) =8+7=15
1.3	50	120	59 + 61 = 120
50	1.3	5/3	32=2^5=[8^(1/3)]^5 =8^(5/3)
1.4	1.4	9804	(100-14)(100+14) =100^2 – 14^2 = 10000-196
		Answer	
2.1	2.1	1/128	(1/2)^7 = 1/128
2.2	2.2	676	One way is to know 25^2=625, then 625+25+26=676
2.3	50	18	The diameter is then 6 which is the diagonal of the square. The area is (1/2)(6)(6) = 18.
50	2.3	-1/3	Adding 180 degrees, reflects the point about the origin. Thus the sine is -1/3
2.4	2.4	216	There are 9 numbers with average (40+8)/2=24. 9(24)=216
		Answer	
3.1	3.1	Yes	2+3=5
3.2	3.2	12	The length can be found by Pythagorean Thrm. L=4, 3(4)=12
3.3	50	63	The sum of the powers of two will be one less than the next power of two. 2^n – 1 = (2-1)(2^(n-1)+2^(n-2)+…+2+1)
50	3.3	121	3^5 – 1 = (3-1)(3^4+3^3+3^2+3+1) (243-1)/2=121
3.4	3.4	144	There are four subjects and 4!=24 ways to arrange them. The math books can be arranged in 3!=6 ways within their group. 6(24) = 144
		Answer	
4.1	4.1	1/3	(4-2)/(10-4)=2/6=1/3
4.2	4.2	60	5!/2! = 120/2 = 60
4.3	50	$\sqrt{3}$	The altitude forms a 30-60-90 triangle with base 1 so the height is $\sqrt{3}$ and the area is (1/2)(2) $\sqrt{3}$ = $\sqrt{3}$.
50	4.3	1/2	4^(-1/2) = 1/(4^1/2) = 1/2
4.4	4.4	101, 103	Try them, most likely error is 89, 91 but 91=7*13.