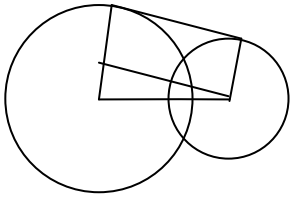
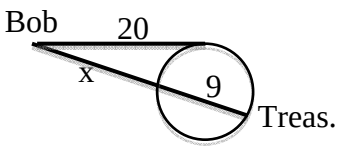
Mental Math

9/11	Answer	Solution
1	1024	Same as two to the tenth.
2	52	This is just the sum from 3 to 10 = (8/2)(13) = 52
3	120	Just need 3*5*8 = 120.
4	160 [degrees]	The exterior angle is 360/18=20, so the interior is 180-20=160.
5	4/3	Log base 8 is (1/3) log base 2. So log_8 16 = (1/3) log_2 16 = 4/3.
6	9	x^2 + y^2 – 2xy = (x-y)^2 = (-3)^2 = 9
7	34 [mph]	[2(40) + 3(30)] / 5 = 34
8	IV or 4^{th}	The vertex is at (-2,-3) and y-intercept is at (0,1); so it misses the 4^{th} quadrant.

Individual Test

9/11	Answer	Solution
1	5/14	6x-3+3=20x-10+5. 6x=20x-5, x=5/14
2	CAB [in that order]	A is a little more than 3, B is 17 and C is negative so CAB.
3	1/8	$2^{-3}=\frac{1}{2^3}=\frac{1}{8}$
4	3	f(x)=5x-4 = 11, 5x = 15, x=3.
5	16	6 + 12 / 3 – 2 (2 – 5) = 6 + 4 – 2(-3) = 10+6 = 16
6	negative	Negative correlation means than one variable goes down as the other goes up. In the graph as x increases, y decreases.
7	-96	Geometric sequence with common ratio=-2. 48 * (-2) = -96.
8	[x=] 5, -4 [either order]	$x^2-x-20=0,(x-5)(x+4)=0$ x={5,-4}
9	32 [cm^2]	A square is a rhombus with its diagonals equal and perpendicular to each other. The area is then (1/2)(8)(8) = 32.
10	37	Four numbers average to 25 so they must total to 100. The three known values 20, 17, and 26 total to 63. The value of x is then 100-63=37.
11	24 [un]	Double a 5 – 12 – 13 pythagorean triple. The other leg is then 2(12) = 24.
12	71 [°]	180 – 34 – 75 = 71
13	22 [in]	$121\pi=\pi r^2,\ r^2=121,\ r=11,\ d=2r=22$
14	24π [ft^3]	$V=\frac{1}{3}\pi r^2 h=\frac{3^2(8)\pi}{3}=24\pi$
15	5	The equation is in slope-intercept form.
16	9 [pizzas]	First subtract the delivery fee and then divide by the number of pizzas. (73.83-10)/6.5 = 9.82 so 9 pizzas. All you really have to do is see that it is less than 10 and more than 9.

9/11	Answer	Solution
17	B	At x=0, A and C are both 1, so the answer must be B.
18	B	A is the converse and C is the contrapositive. B is the inverse.
19	True	The inverse is the contrapositive of the converse and thus its logical equivalent. For the statement p->q, this implication is true in all cases except when p is true and q is false. Converse is q->p Inverse is ~p -> ~q Contrapositive ~q -> ~p
20	3/2	Using the properties of logs: $\ln e = 1,\ \ln x^y = y \ln x$ this reduces to: $\frac{5}{a} + \frac{24}{ab} - \frac{15}{3a} = 16,\ 15b + 72 - 15b = 48ab,\ ab = \frac{72}{48} = \frac{3}{2}$
21	30 [degrees]	AD is a diameter so angle ABD is 90 degrees. Since angle DAB is 60 degrees, angle ADB measures 30 degrees.
22	7/25	You don't need all these facts. P(A or B) = P(A) + P(B) – P(A and B) = 20 + 16 – 8 = 28% = 7/25
23	56 [in]	The area is 12 so we have to consider dimensions of rectangles that multiply to 12. 1x12, 2x6, 3x4; giving perimeters of 26+16+14 = 56.
24	8 [integers]	Written as a single inequality: $-11 \le 3x - 7 \le 11$ $-4 \le 3x \le 18$ $-\frac{4}{3} \le x \le 6$ x=-1, 0, 1, 2, 3, 4, 5, 6 so the number is 8.
25	257	5 | 235, 3 | 237, 11 | 253, 257 is prime.
26	875/20736	Emma (E) wins a round with probability 21/36=7/12 and Francis (F) with probability 15/36=5/12. If Francis wins in 4 rounds then we need P(EFFF) = (7/12)(5/12)(5/12)(5/12) = 875/20736.

9/11	Answer	Solution
27	35 [in]	 The figure to the left gives an idea of the solution method. The quadrilateral is a rectangle with sides 14 and x (the length of the tangent). This makes the triangle right with dimensions 37, 26-14=12, and x. Solving for x gives 35.
28	[$] 180	If there are n people who paid c dollars then: nc = (n+2)(c-2) = (n-5)(c+6) which yields two equations for n and c. n is 15 and c is 12. Dinner costs nc = 15(12) = $180.
29	108 [ways]	Put the two people who won't sit in the second chair into the first and the one person who refuses to sit in the first chair into the second. There are then 3 people left who can occupy the remaining seat in the first and the other two will sit in the second. Now there are 3!=6 ways to interchange each chair so the total is 3*6*6 = 108 ways.
30	25 [ft]	$20^2 = x(x+9)$ $(x+25)(x-16) = 0$; x+9 = 25 feet total $x = 16$ 
31	40 [ways]	First consider the case where the players in each sport are indistinguishable. Take A A B B C C and put them around a table so no two of the same letters are next to each other. Taking rotational symmetry into account there are only two configurations for the A's. AxAxxx and AxxAxx. There are a total of 5 unique cases. ABACBC, ACABCB, ABCABC, ACBACB, ACBABC. Now since the players in each sport are distinguishable, we can interchange them to make a new set, so the answer is 5x2x2x2 = 40

9/11	Answer	Solution
32	11/29	Since P is an even function only even exponents can have non-zero coefficients, so a=0. $P(2) = 16 + 4b + c = 1$ $P(3) = 81 + 9b + c = 11$ $b = -11,\ c = 29$
33	55	$E = (1)\frac{1}{55} + (4)\frac{2}{55} + (9)\frac{3}{55} + \cdots + +(100)\frac{10}{55}$ $= \frac{1}{55}\left[1^3 + 2^3 + \cdots + 10^3\right] = \frac{1}{55}\left[\frac{10(11)}{2}\right]^2 = 55$
34	45 [°]	A least a couple of ways to do this. (1) Translate each line to pass through the origin and then locate vectors on the lines i+7j and 4i+3j. The cosine of the angle between them is (4+21)/[root(50)root(25)] = 1/root(2). The angle is 45 degrees. One could also use the formula for the tangent of the angle between the curves in terms of slope. tan(a)=(m1-m2)/(1+m1m2) = (7-3/4)/(1+7(3/4)) = 1, and again the angle is 45 degrees.
35	12 [un^2]	The ellipse can be parameterized by x=2cos(t) and y=3sin(t). For any value t, the rectangle has area (2x)(2y) = 4*6sin(t)cos(t) = 12sin(2t). The sine function has a maximum of 1, so the maximum area is 12.

9/11	Answer	Solution
36	45 [cm]	From the vertex of the rectangle draw a horizontal line to directly above the center of the circle and then another down to the center and back to the vertex. This forms a right triangle with legs: R-9 and R-18 and hypotenuse R. Pythagoras gives us an equation to solve for R. $(R-9)^2+(R-18)^2=R^2$ Since R>9, R must be 45 cm. $(R-45)(R-9)=0$
37	72	There are a couple of methods. One can set up 3 equations using the three points, ie. 2A+1B+1C=D, etc. and solve for A,B,C in terms of D and then choose D so that it contains the point (20,0,3). Alternately, one can solve for the normal vector using the determinant of the matrix [(i,j,k),(2,3,-3),(0,-2,1)] where the last vectors were formed by subtracting the first point from the other two. The determinant gives the equation of the plane as 3x+2y+4z=12. Fitting a parallel plane to the new point (20,0,3) gives D=72.
38	2 [un^2]	In rectangular coordinates x=rcosA, y=rsinA. $\left(-\sqrt{3},1\right),\left(-1,\sqrt{3}\right),\left(1,\sqrt{3}\right)\left(\sqrt{3},1\right)$ is a trapezoid with base 2 and $2\sqrt{3}$ and height $\sqrt{3}-1$. The area is $\frac{1}{2}\left(\sqrt{3}-1\right)\left(2+2\sqrt{3}\right)=\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)=2$

9/11	Answer	Solution
39	200	Let the small radius be r, the height h and large radius, R. If H is the height of the overall cone then by similar triangles H/R = (H-h)/r which can be solved for H. The formula for the volume of the frustrum is then: $V = \frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$. We have h=2r and R=2h=4r. $V = 350\pi = 14\pi r^3$, $r(2r)(4r) = 8r^3 = 350 * 8 / 14 = 200$
40	40 [ways]	By brute force: 5 – 4 – 3 … 6 ways 5 – 4 – 0 … 4 ways 5 – 3 – 1 … 6 ways 5 – 1 – 0 … 4 ways 4 – 3 – 2 … 6 ways 4 – 2 – 0 … 4 ways 3 – 2 – 1 … 6 ways 2 – 1 – 0 … 4 ways Total 40 ways

Multiple Choice

9	11	Answer	Solution
1	1	B	A rhombus is constructed, all sides equal, which has the property that the diagonals bisect the angles. (B)
2	2	C	40% of the circle is the same as 60% of the triangle. (2/5)x = (3/5)15 = 9, x=45/2
50	3	D	$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}=\begin{bmatrix}2 & 7\\ 0 & 3\end{bmatrix}$ and the sum is 2+7+0+3 = 12 (D)
4	4	E [64π]	The dimensions are 8 – 12 – 20. Thus the largest sphere has radius 4. Surface area is $SA = 4\pi r^2$ or 64π (E)
5	5	A	Player A wins only in the situation where the first three flips are heads, B will win in all other circumstances.
50	6	B	The '2' in cos2x does not contribute to the answer. The maximum is RSS (root sum of squares) of the amplitudes: $\sqrt{2^2+1}=\sqrt{5}$ (B)
50	7	D	The sum of all the solutions is 0 (-b/a). The only real valued solution is -1 so the others must add to 1 (D).
8	8	D	7(11) – 7 – 11 = 59 (D).
9	9	A	$s=\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\cdots$ $\frac{1}{3}s=\frac{1}{9}+\frac{2}{27}+\frac{3}{81}+\cdots$, subtract $\frac{2}{3}s=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1/3}{1-1/3}=\frac{1}{2}$ (A) $s=\frac{3}{4}$
10	10	C	MIC = (M+I+C)(M+I+C+1), the product of two consecutive integers. Some trial and error shows that MIC=156 = 12(13). (C)
3	50	A	The x-coordinate is –b/2a = -2 or one can complete the square $y=2x^2+8x-11=2(x+2)^2-8-11$, both give a y-coordinate of -19.

9	11	Answer	Solution
6	50	B	The "shoelace" method can be used or the triangle can be put inside a square (2,1), (2,7), (8,7) and (8,1) and subtract the area of the three extra right triangle. 6(6)-3-12-9 = 12 (B)
7	50	B	$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+ac+bc}{abc}$ which is just the sum of the products taken two at a time, 2, over the product. Also, one could construct the polynomial $1-3x+2x^2-6x^3$ which has roots that are the reciprocals of the original polynomial (B).

Team Test

G1	G2	Answer	Solution
1	1	54	$A = \frac{4(6^3)\pi}{3}, C = \frac{(6^3)\pi}{3},$ $B = 6^3,\ BC / A = 6^3 / 4 = 216 / 4 = 54$
20	2	143/16	The first term is just 4, the second is 1/(1-4/5)=5 as the sum of an infinite geometric series and the third has a lot of terms cancel to give 1/16. The answer is then 4 + 5 – 1/16 = 143/16.
3	3	21 [ways]	Add three extra pieces of candy so that everyone is assured of a piece and then line up the pieces: 1, 2, 3, 4, 5, 6, 7, 8. Dividing the candy is the same as choosing 2 of the spaces between the candies. There are 7 choose 2 = 21 ways to do this. Now take a piece of candy back from each kid.
4	4	4	$17^2 = 289 \equiv 3 \bmod(11)$, *so* $17^8 \equiv 3^4 = 81 \equiv 4 \bmod(11)$
5	5	495	There are 9000 4-digit numbers (9*10*10*10). There are 8*9*9*9 with no 4's and 1*9*9*9 where only the first digit is 4; 8*1*9*9 for the 2nd digit, etc. 9000-8*9*9*9-3(8*9*9) = 495.

G1	G2	Answer	Solution
20	6	$14168_{[11]}$	In base 10 : 3(4)(5)(6)(7)(8) = 20160. One can change to base 11 by repeated division by 11. $11\overline{)\,20160}$ r8 $11\overline{)\,1832}$ r6 $11\overline{)\,166}$ r1 $11\overline{)\,15}$ r4 $11\overline{)\,1}$ r1 0
7	7	2	By substituting for the repeating terms, we get: $x = \dfrac{1}{1-\dfrac{5}{8+x}}$ or $s^2+2s-8=0$ which has solutions 2 and -4.
20	8	496	The known perfect numbers are even of the form $2^{p-1}(2^p-1)$ where 2^p-1 is prime. p=2 gives the perfect number 6, p=3 gives 28 and p=5 gives 496.
9	9	$64\pi^2$ [mm^2]	A form of Pappus' theorem can be used. Or cut the ring and straighten it out to make a cylinder (approx). The surface area is the circumference of the base times the height. The base has circumference $2\pi(2)$ and the height is the circumference of the circle from the center of the base $2\pi(8)$. Multiplying, one gets $64\pi^2$ sq mm.

G1	G2	Answer	Solution
10	10	67/288	Draw a square where each side starts at 1:00pm and in 5 minute increments goes to 2:00pm. Put Stacey' arrival on the vertical and Trung's on the horizontal. If Trung arrives at 1:00, Stacey can arrive between then at 1:10 and they meet. For every minute after 1:00 that Trung arrives, Stacey can arrive a minute later, so draw a 45-degree line. Similarly, if Stacey arrives at 1:00 they will meet as long as Trung arrives by 1:05 and there is a second 45-degree line. Anywhere in the middle of the square and they meet, otherwise they miss. The probability is then the are of the stripe over the area of the square. (288-100-121)/288 = 67/288.
2	20	11	Since $f^{-1}(f(x)) = f(f^{-1}(x)) = x$, $g(x) = f(x)$ g(3) = f(3) = 11
6	20	32 [ways]	At each intersection count the number of ways to get to your friends house. There will be 1's across the top and down the right column. For the other intersections, they will be the sum of the counts at the intersection to the right and the one directly above it. If there is a gap, don't include it. By this recursion, there are 32 paths to your friends house.
8	20	$\frac{27\pi}{2}$ [un^2]	Completing the square we have: $\frac{(x-6)^2}{9^2}+\frac{y^2}{3^2}=1$ and the total area is $ab\pi = 3(9)\pi = 27\pi$. Since one axis is on the x-axis, the desired area is half of this.

Pressure Round

9	11	Answer	Solution
1	1	$\frac{111}{512}$	There must be a better way but they could be enumerated. HHHHxxxxx = 32 ways THHHHxxxx = 16 ways xTHHHHxxx = 16 ways xxTHHHHxx = 16 ways xxxTHHHHx = 16 ways xxxxTHHHH = 15 ways (don't double count xxxx=HHHH) Total 111 out of 2^9=512 ways.
9	2	$(-\infty,6]$	The parabola has an axis of symmetry at b=1, so a minimum of 5 and a maximum of infinity. The absolute value thus doesn't have any effect.
3	3	2162	Find two largest squarefree integers less than 50. 49 = 7*7 and 48 = 4*4*3 so we use 46 and 47 as 46+47=93=3*31 and 46(47) = 2162.
9	4	0	For every non-zero positive value of y, there is a corresponding negative value. Alternately, the points will be the 9th roots of 1, solving x^9-1 = 0. The roots sum to 0.
5	5	2,100,010,006	There are going to be a lot of 0's so there must be more than one 1. Try two 1's, then one 2. Filling with 0's yields the answer 2,100,010,006.
2	9	8,388,608 [bits]	2^10 = 1024. 8*1024*1024=8388608
4	9	$\frac{1}{24}$	$1=\log_w * \log_x y * \log_y z * \log_z w = 24wxyz$ so wxyz = 1/24.

College Bowl Round 1

9	11	Answer	Solution
1	1	14	(1/3)(1/2)(84)=14
2	2	20 [yrs old]	x+x-6=24, x=15, x+5=20
50	3	14	F=4, E=6, V=4
4	4	4	1x2x3x4x6 = 24*6 ends in a 4.
50	5	15/4	20 = 4y+5
6	6	5/7 [miles]	5[4/7 – 3/7]
7	7	36 [°]	Interior = 108. Each of the outer triangles are isosceles, base angle=36.
8	8	3	3, 4, 2, -3, -5, -2, 3
50	9	3/2	sin^2(15)+2sin(15)cos(15) +cos^2(15) = 1+sin(2*15)= 1+(1/2)=3/2
10	10	32 plus pi [ft^2]	4 sections of 8x1 and 4 quarter circles of radius 1.
3	50	26	F=6, E=12, V=8
5	50	28	1+2+3+4+6+12
9	50	5	The hypotenuse of the right triangle is a diameter.

College Bowl Round 2

9	11	Answer	Solution
1	1	150 [°]	The angles are 30-60-90
2	2	4209	61(33)+36(61)=61(69) [6*7][1*9]=4209
50	3	150 [°]	Sine is the y-coord so sin30 = sin150.
4	4	8 times the square root of 6	384=64*6
50	5	8 or -8 [either one]	Square root of 2 times 32. -8 could happen with a common ratio of 2i.
6	6	32 pi [in^2]	2(2^2)pi + 6*2*2*pi = 32pi
7	7	0	If x works then –x also works.
8	8	1/2	Ignore all the rest of the numbers, there are only two outcomes that matter.
50	9	9	9^(3/2) = 27
10	10	150	180=(3^2)(2^2)(5) so we need to multiply by 3*2*5^2=150
3	50	46 [%]	[.4*20+.5*30]/50= 23/50.
5	50	-8	21/3 – 3*5 = -8
9	50	8	72=(2^3)(3^2) so 4 choices for 2 and 2 for 3. 4(2)=8.

College Bowl Round 3

9	11	Answer	Solution
1	1	92	-8+(21-1)*5=92
2	2	3/8	4C2/16 = 3/8
50	3	156 [°]	180 – 360/15
4	4	1 / pi [un^2]	2pi(r)=2; r = 1/pi. A=pi(r^2)= pi (1/pi^2) = 1/pi.
50	5	2	1/log4(2) = log2(4)=2
6	6	255	4^4 - 1
7	7	5/12	P(7)=1/6. [1- 1/6] / 2 = 5/12
8	8	89	Has to end in 9.
50	9	2 root(10) / 7	Half the base is 3 which means the height is root(49-9)=2root(10). sine is opp/hypot.
10	10	8	2y+1=3, y=1. So, f(3) = f(2(1)+3)) = 1^2 + 6(1) + 1 =8.
3	50	120 [°]	180 – 360/6
5	50	31	(3^3)+(4^2)-12 = 27+16-12 = 31
9	50	1	A=P/2 (r). 6 = 12/2 r. r=1

College Bowl Round 4

9	11	Answer	Solution
1	1	33	50/2 + 50/3 – 50/6 = 25+16-8=33
2	2	275	5(1+2+…+10)= 5(10)(11)/2=275
50	3	1	y=(x+1)(x-2)^2 + 6 Without the +6, it crosses at -1 and touches at 2. +6 means it only crosses at -1.
4	4	4	(0,9), (9,0), (-9,0), (0,-9)
50	5	Square root of 37 [un]	Law of cosines.
6	6	39,975	(200-5)(200+5) = 40000 – 25 = 39975
7	7	12	Eulers method. 708-672=36. 672=36*18+24 and gcf(36,24)=12.
8	8	1/3	b^2-4ac = 4-4(3)(m) = 0. m = 1/3
50	9	3/7	Cannot draw any pennies, there are 4C2=6 ways of drawing at least 15 cents. Since we know two pennies were not drawn, there are 6C2 – 1 = 14 possible ways. 6/14=3/7
10	10	19/9	The regions have areas 1pi, 3pi and 6pi. The expected value is 5(1/9)+ 3(3/9)+1(5/9)=20/9
3	50	24 [cm^2]	The diagonals form four 3-4-5 right triangles. 2(3)(4)=24
5	50	(0 , 8) or 0 comma 8	Plug in x=0.
9	50	2/5	Cannot draw any pennies and there are four other coins. (4/6)(3/5) = 2/5.

College Bowl Round 5

9	11	Answer	Solution
1	1	59 and 61, both are needed either order.	51, 57 are not prime.
2	2	2+5+23 or 2+11+17 [either one]	One number has to be 2.
50	3	9 pi/2	Each hour is 2pi so 2 $\frac{1}{4}$ hours = 9pi/2.
4	4	Square root of 5 [cm]	Unfolding the cube, Alice travels the hypotenuse of a 1-2-root 5 triangle.
50	5	2 minus x	log 50 = log (100/2) = log 100 – log 2 = 2-x
6	6	8	(10+1)^8 = 10^8 + …+8(10)(1)+ 1 = …81.
7	7	3/4	With two tosses, only HH fails.
8	8	1024	Factoring a two out of each number in the numerator, we get 2^10*10!/10! = 2^10 = 1024.
50	9	Square root of 13.	sqrt(2^2 + 3^2) = sqrt(13).
10	10	0	Multiplying by x^2, you get x^4 – 2x^2 + 1=0. Sum of roots is –b/a = 0.
3	50	180 [°]	W=Z and X=Y and all four add to 360.
5	50	35 [°]	(180-110)/2
9	50	3 comma 3 times square root of 3. $(3,3\sqrt{3})$	x=6cos60=3; y=6sin60=3root(3)

College Bowl Round 6

9	11	Answer	Solution
1	1	90.16	9(10) followed by 2(8) and adjusted for two decimals.
2	2	24 [ways]	There are 4 other letters, 4! = 24.
50	3	Hyperbola	definition
4	4	5/16	1, 0, 1/2, 1/4, 3/8, 5/16
50	5	-2 and -1 [any order]	F(x)=(x+2)/(x^2+3x+2) =(x+2)/[(x+2)(x+1)]. Even though x+2 divides, -2 is still not in the domain.
6	6	17	(6- -2)^2+(-12 – 3)^2 = 8^2 + 15^2 = 17^2. Answer is 17.
7	7	1	13^5 = (12+1)^5 when expanded only the last term is not divisible by 3. Ans 1
8	8	72 [min]	(1/2)+(1/3) = [1/(6/5)]. 6/5 hours = 72 minutes
50	9	6 pi [un^2]	Divide by the constants, square and then add the equations. (x/3)^2+(y/2)^2= sin^2t+cos^2t = 1. Figure is an ellipse with a=3, b=2. Area = (3)(2)pi = 6 pi.
10	10	2	S=(1/2)+(2/4)+(3/8) +…. (1/2)S=(1/4)+(2/8) +… Subtract (1/2)S=(1/2)+(1/4)+… = 1, so S=2.
3	50	56	(8*7*6)/(3*2*1) = 8*7 = 56.
5	50	108 pi [m^2]	[4(pi)r^2]/2+(pi)r^2 =[2*36+36]pi = 108pi
9	50	21 [un^2]	The triangle has base and height equal to 6 and 7. Area = (1/2)(6)(7) = 21.

College Bowl Extra

9	11	Answer	Solution
1	1	256	80 / 2 / 5 = 8, rectangle is 8x32. Area = 256.
2	2	4 , 3	(2*2+8)/3 comma (2*4+1)/3 = (4,3)
3	3	384	4! ways of interchanging the couples and 2^4 ways of ordering the husbands and wives. 4!*2^4 = 384.
4	4	3	Ignore all but the highest order, (12x-10)/(4x+20) -> (12x)/(4x) = 3.