"Math is Cool" Championships -- 2020-21

High School

Mental Math Solutions

	Answer	Solution
1	66	What is the product of three-elevenths and two hundred forty-two? $$\frac{3}{11}\cdot\frac{242}{1}=66$$
2	3	What is the slope of the line connecting the point four comma five and the point one comma negative four? $$m=\frac{5-(-4)}{4-1}=3$$
3	81 [units squared]	What is the area of a square with perimeter of thirty-six units? **Side = 9, area = 9^2 = 81**
4	6	When two standard six-sided dice are rolled, the probability that the sum of the numbers in the top faces is less than 5 is equal to x. What is the reciprocal of x? **P(sum < 5) = 6/36 = 1/6**
5	55	What is the tenth term in the Fibonacci sequence? Assume the first two Fibonacci numbers are both equal to 1. **1, 1, 2, 3, 5, 8, 13, 21, 34, 55**
6	119	If 289 over x equals x over 49, what is the largest possible value of x? $$\frac{289}{x}=\frac{x}{49}$$ **x^2 = 289·49** **x = 17·7 = 119**
7	512	What is the area of the triangle bounded by the lines x = 0, y = 0, and x plus 4 y equals 64? Find the x- and y-intercepts for the last line. They are (0, 16) and (64, 0). Then find the area of the triangle: 16 * 64 / 2.
8	49	When written in base ten, how many zeroes are at the end of two-hundred-factorial? Trick: 200 / 5 = 40, 200 / 25 = 8, and 200/125 = 1 (+ some remainder). 40 + 8 +1 = 49.

"Math is Cool" Championships -- 2020-21
High School
Individual Test Solutions

	Answer	Solution
1	186	Evaluate: $-2+8/4+3*(17+9*5)$ -2 + 2 + 3*(62)
2	5	Two whole numbers have a sum of 11 and product of 24. What is the positive difference between the numbers? x + y = 11 xy = 24 x(11 – x) = 24 $x^2 - 11x + 24 = 0$ (x – 3)(x – 8) = 0 x = 8, 3
3	3875	Evaluate: $25^2+35^2+45^2$ **625 + 1225 + 2025 = 3875**
4	480 [apples]	If there are 6 apples in a bin, 4 bins in a bundle, and 2 bundles in a crate, how many apples does 10 crates contain? $6\times4\times2\times10=480$
5	4096 $[\mathrm{m}^2]$	What is the area, in square meters, of a rectangle having width of 32 meters and height of 128 meters? $A=lw=32*128=2^5*2^7=2^{12}=4096$
6	2	Find the tens digit of 5^{2020}. For $n>1$, 5^n always ends in 25.
7	120	If 108 is 90% of x, find the value of x. $\frac{108}{x}=\frac{9}{10}$
8	2	Find the slope of a line perpendicular to $x+2y=8$. The line has slope $-1/2$, taking negative reciprocals yields 2.
9	3	How many positive multiples of 9 have three digits, all of which are the same? Multiples of 9 have the sum of their digits divisible by 9. So the multiples we want are 333, 666, and 999.

10	53280 [minutes]	How many minutes are in 37 days? $$37 * 24 * 60 = 53280$$
11	4 [pennies]	Richard has 9 coins consisting only of pennies and nickels. If the total value of the coins is $0.29, how many pennies does Richard have? Solve the system $$p + n = 9$$ $$p + 5n = 29$$
12	25	Find the sum of the infinite geometric series: $10 + 6 + 3.6 + \cdots$ $$S = \frac{a}{1-r} = \frac{10}{1 - \frac{6}{10}} = 25$$
13	60 [mph]	A car traveled 281 miles in 4 hours and 41 minutes. What was the speed of the car in miles per hour? 4 hours and 41 minutes totals up to 281 minutes, so the car was going 1 mile per minute, or 60 miles per hour.
14	124 [degrees]	Angles A and B are supplementary. If $m\angle A = 56°$, find $m\angle B$, in degrees. $$180 - 56 = 124$$
15	109	The midpoint of the line segment connecting $(2, 7)$ and $(-8, 13)$ is (a, b). Find the value of $a^2 + b^2$. Taking the average of each coordinate, midpoint is $(-3, 10)$.
16	6	Find the remainder when $x^{2020} + 2x + 3$ is divided by $x - 1$. Evaluate the polynomial at $x = 1$.
17	6	What is the median of the first six smallest positive prime numbers? The set {2, 3, 5, 7, 11, 13} has an even number of elements, so the median is the average of the middle pair, or 6.
18	-15	Find the smallest solution to the equation $x^2 + 12x = 45$. Factoring, we get $$(x + 15)(x - 3) = 0$$ The smallest root is $x = -15$.
19	14	If $x■y = x + 3y - 1$ for all integers x and y, evaluate: $(1■2)■3$ We have $1■2 = 6$ and $6■3 = 14$

20	15	What is the greatest common factor of 45, 60, and 75? Since $45 = 15 * 3$, $60 = 15 * 4$, and $75 = 15 * 5$, and 3, 4, and 5 have no other common factors, the answer is 15.
21	6 [sides]	Exactly three of the interior angles of a convex polygon are obtuse. What is the largest number of sides this polygon can have? Let n be the number of sides of the polygon, with 3 obtuse angles and $n - 3$ acute angles. An upper bound for the sum of the three obtuse angles is $3 \times 180 = 540$. An upper bound for the sum of the other $n - 3$ angles is $(n - 3)(90) = 90n - 270$. So an (unattainable) upper bound for the sum of all the angles is $540 + (90n - 270) = 90n + 270$. But we know the sum of all the interior angles of the polygon is equal to $180(n - 2)$. Thus, we have the inequality: $$180n - 360 < 90n + 270$$ Solving, we get $n < 7$
22	900	The first four terms of an arithmetic sequence are $x + 1$, $2y$, $3x + y$, and 12. What is the sum of the first 24 terms of this sequence? In an arithmetic sequence, twice the middle term is equal to the sum of the adjacent terms. Using the third term as the middle, we have $2(3x + y) = 2y + 12$, or $x = 2$. Using the second term as the middle, we have $2(2y) = (x + 1) + (3x + y) = 9 + y$, or $y = 3$. The sequence is 3, 6, 9, 12, … The 24th term is $3 + 23(3) = 72$ and the desired sum is $\frac{24}{2}(3 + 72) = 900$.
23	17	The arithmetic mean of the following set $$S = \{1, 2, 4, 8, 9, 10, 14, 16, 17\}$$ is x. If a single element is removed from S, the arithmetic mean of the resulting set is $x - 1$. Which number was removed? The arithmetic mean of S is 9, so the arithmetic mean of the resulting set is 8, meaning the numbers add up to 64. Thus, $81 - 64 = 17$ was removed.
24	24	Find the area of a triangle with vertices at $(1, 3)$, $(7, -8)$, and $(1, -5)$. Use Shoelace Method, graphing, or whatever your favorite technique is for solving these problems.
25	25	For which positive integer n is $5^{50} = n^n$ true? $$5^{50} = 5^{2\times 25} = (5^2)^{25} = 25^{25}$$

26	4	For how many integer values of x is $\sqrt{12-\sqrt{x}}$ also an integer? Guess-and-check perfect squares to get $x \in \{9, 64, 121, 144\}$, so 4 values.
27	30 [degrees]	The interior angles of a hexagon are in the ratio $3:4:4:4:4:5$. How many degrees are in the measure of the smallest exterior angle? Solving $3x + 4(4x) + 5x = 720$ to get $x = 30$. The largest interior angle is $5 * 30 = 150$, making the smallest exterior angle $180 - 150 = 30$.
28	1365	How many even integers between 1 and one million have digits that are all primes? Digits must be 2, 3, 5, or 7. The last digit must be 2. There can be at most 6 digits, making the total count equal to: $1 + 4^1 + 4^2 + 4^3 + 4^4 + 4^5 = 1365$
29	533	A right circular cylinder is 3 feet tall and has a base 16 feet in diameter. Adding x feet to either the cylinder's radius or height (while keeping the other dimension constant) increases the volume of the cylinder by the same nonzero amount. Find the largest integer less than $100x$. The problem translated to equations is: $$\pi 8^2(3+x) = \pi(8+x)^2(3)$$ This equation has solutions $x \in \left\{0, \frac{16}{3}\right\}$. The solution $x = 0$ does not make sense in this context, so the answer is the largest integer less than $100\left(\frac{16}{3}\right) \approx 533.33$ ..., or 533.
30	217	If $r = \sqrt{\frac{1}{25} + \frac{1}{144}}$, find the smallest integer greater than $1000r$. We have $r = \sqrt{\frac{169}{25 \cdot 144}} = \frac{13}{60} \approx .21667$, so $1000r \approx 216.67$, making the answer 217.

31	173	Stacey chooses four distinct numbers at random, without replacement, from the set of integers 1 to 10, inclusive. The probability the product of these four numbers is divisible by 10 is equal to a/b, where a and b are positive integers with no common factors. Find $a+b$. For the product to be divisible by 10, either: Case I – One of the numbers is 10. This is $\frac{C(9,3)}{C(10,4)} = \frac{84}{210}$ Case II – One of the numbers is a 5, and there's at least one even number (that's not 10): $\frac{C(4,1)C(4,2)}{C(10,4)} + \frac{C(4,2)C(4,1)}{C(10,4)} + \frac{C(4,3)C(4,0)}{C(10,4)} = \frac{52}{210}$ Total probability is $\frac{84}{210} + \frac{52}{210} = \frac{136}{210} = \frac{68}{105}$, so $a+b=173$.
32	1777	If x, y, and z are positive integers such that $1/x + 1/y + 1/z = 6/7$, evaluate: $x^2+y^2+z^2$ WLOG, assume that $x \le y \le z$. Since the fractions add up to something close to 1, the denominators would need to be relatively small. Moreover, since $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, this is a good initial guess since $\frac{1}{z} = \frac{6}{7} - \frac{5}{6} = \frac{1}{42}$, making $z = 42$. One can dig into some more inequalities to show that this solution is unique. Answer is: $2^2 + 3^2 + 42^2 = 1777$
33	13	Let $\sqrt{10!} = a!\sqrt{b}$, where a and b are integers and b is prime. Find $a+b$. $\sqrt{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 720\sqrt{7} = 6!\sqrt{7}$, so $a+b=13$
34	15	Find the product of the *real* roots of the equation: $$x^4 - 12x^3 + 60x^2 - 164x + 195 = 0$$ By the Rational Root Theorem, none of the roots are even. Using guess-and-check along with synthetic division, we find that 3 and 5 are two real roots and the remaining polynomial factor is $x^2 - 4x + 13$. The discriminant of this quadratic is $(-4)^2 - 4(1)(13) = 16 - 52 < 0$, so it can't have any real roots. The desired product is $3 \times 5 = 15$.

35	250	Suppose f is a function such that $f(xy) = f(x)/y$ for all positive reals x and y. If $f(50) = 300$, find the value of $f(60)$. Since $60 = 50\left(\frac{6}{5}\right)$, we have $$f(60) = f\left(50 \cdot \frac{6}{5}\right) = \frac{f(50)}{6/5} = \frac{5}{6}f(50)$$ Thus, the answer is $\left(\frac{5}{6}\right)300 = 250$
36	142857	If the letters of the word ELEMENT are randomly arranged, the probability the three Es are consecutive is equal to p. Find the largest integer less than $p \times 10^6$. Total arrangements without restriction is $\frac{7!}{3!}$. If we treat the three consecutive Es as one unit, it's basically the same as arranging ELMNT, so 5! ways. Thus, the probability is $$p = \frac{5!}{7!/3!} = \frac{1}{7} = .\overline{142857}$$ The answer follows, as $p \times 10^6 \approx 142857.1$
37	210 [sq. units]	What is the area of a triangle with side lengths 17, 28, and 39? Semi perimeter is $\frac{17+28+39}{2} = 42$; by Heron's Formula: $$\sqrt{42(42-17)(42-28)(42-39)} = \sqrt{(2)(3)(7)(5)(5)(2)(7)(3)} = (2)(3)(7)(5) = 210$$
38	160	Zozo is a city where three-fifths of the residents will always lie, while the rest always tells the truth. Locke asked three randomly selected people from Zozo if it rained yesterday and all of them said yes. The probability that it truly rained yesterday is equal to p. Find the value of $700p$. Either all three people were liars or all three were truth-tellers. Thus, the total probability is $.6^3 + .4^3 = .28$. If it was indeed raining, we want the all truth-teller case, or $.4^3 = .064$. Thus, the answer is $700p = 700\left(\frac{.064}{.28}\right) = 160$.

39

25

The least common multiple of the consecutive integers from 20 to k, inclusive, is greater than 10^6. If $k > 20$, what is the smallest possible value of k?

We have $20 = 2^2 \times 5$; onward, we work through the successive integers and see what new factors they bring to the table. We stop when the cumulative product exceeds 10^6:

n	Product of New Factors Introduced	Cumulative Product
20	20	20
21	3×7	420
22	11	4620
23	23	106260
24	Adds another 2	212520
25	5	1062600

The expectation is that the cumulative products in the third column be estimated, rather than computed directly.

40

25

An ellipse in the xy-plane has foci at $(3, 4)$ and $(10, 20)$. Moreover, the ellipse is tangent to the x-axis. Find the length of the major axis of the ellipse.

Let $F_1 = (3, 4)$ and $F_2 = (10, 20)$. Reflect F_2 across the x-axis to obtain $F_2' = (10, -20)$. The line segment connecting F_1 and F'_2 will pass through the point of tangency of the ellipse and x-axis; call this point P. The length of the major axis of the ellipse is equal to the constant sum of the focal distances from a point on the ellipse, or $F_1P + F_2P$. Since $F_2P = F_2'P$ due to reflections, $F_1P + F_2P = F_1P + F_2'P = F_1F_2'$. By the distance formula, the answer is

$$\sqrt{(3-10)^2 + \left(4-(-20)\right)^2} = 25$$

41

28 [digits]

How many digits does $4^{16}5^{25}$ have when multiplied out expressed in base 10?

We have $4^{16}5^{25} = 2^7 10^{25}$, or $2^7 = 128$ with 25 zeroes at the end, giving a total of $3 + 25 = 28$ digits.

42	204	How many positive integers satisfy all three properties? • Has three digits • All digits are distinct • The digits are in increasing order or decreasing order. For example, 458 or 931. If the digits are in decreasing order, then we can choose from all the digits 0 to 9 in $C(10,3) = 120$ ways. If the digits are in increasing order, 0 cannot be a digit, leaving us with $C(9,3) = 84$ ways. In total, there are $120 + 84 = 204$ possibilities.
43	3025	A line bisecting the larger acute angle in a triangle with sides of length 66 cm, 88 cm, and 110 cm divides the opposite side into two segments. The length of the longer segment of that side is L centimeters. Find the value of L^2. Note that the triangle is a 3-4-5 right triangle with a scale factor of 22. Therefore, the larger acute angle is actually the one opposite the side 88 cm long. Let the length of one of the segments equal x. By the angle bisector theorem: $$\frac{66}{x} = \frac{110}{88 - x}$$ Solving, we get $x = 33$. So the longer side $L = 55$ and $L^2 = 3025$.
44	3	If S is the sum of the solutions to $\|y + 3\| = 7 - \|2 - 5y\|$, find the value of $6S$. There are three cases to consider: $$y + 3 = 7 - (2 - 5y)$$ $$y + 3 = 7 - \left(-(2 - 5y)\right)$$ $$-(y - 3) = 7 - (2 - 5y)$$ Only the first two equations yield values that work when plugging into the original equation: $-\frac{1}{2}$, 1. Thus, $S = \frac{1}{2}$ and $6S = 3$.

45	360 [minutes]	Tracy starts a trip sometime between 8 A.M. and 9 A.M., when the hour and minute hands of her watch are together. She arrives at her destination the same day sometime between 2 P.M. and 3 P.M., when the hour and minute hands of her watch are $180°$ apart. How many minutes was her trip? To go 360° around a clock takes 12 hours, so going 180° takes 6 hours.

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High School

Multiple Choice Solutions

9/ 10th	11/ 12th	Answer	Solution
1	**1**	B	Find the perimeter of a rectangle 6 meters wide and area of 24 square meters. A) 25 meters B) 20 meters C) 15 meters D) 10 meters E) Answer not given. Rectangle is 6×4.
2		D	Find the largest integer n where $$n^{200} < 5^{300}$$ A) 8 B) 9 C) 10 D) 11 E) Answer not given. Since n is positive, take the 100th root of both sides to get $n^2 < 5^3 = 125$. Now it's easy to plug in the answer choices to see that the answer is 11.
	2	D	Find the least common multiple of $24x^6y^2z$ and $28x^4y^3$. A) $4x^4y^2$ B) $672x^{24}y^6$ C) $52x^{10}y^5z$ D) $168x^6y^3z$ E) Answer not given. For the numbers, take the regular LCM. For each variable, take the maximum of the exponent.

3	**3**	B	If $\frac{x+y}{2x-y} = \frac{3}{2}$, find the value of $\frac{x}{y}$. A) $4/5$ B) $5/4$ C) $6/5$ D) $5/6$ E) Answer not given. Cross multiply to get $2x + 2y = 6x - 3y$, or $x/y = 5/4$.
4	**4**	A	Which of the following numbers is both complex and rational. A) 3.14159 B) e C) $\sqrt{2}$ D) i E) Answer not given. All of choices A to D are complex numbers; only A is rational, being a ratio of two integers.
5	**5**	E	Let $n > 1$ be the smallest composite number having no prime factors less than 10. Find the sum of the digits of n^2. A) 12 B) 13 C) 14 D) 15 E) Answer not given. The smallest prime greater than 10 is 11. The prime factors aren't necessarily distinct, so conveniently, we arrive at an answer of $n = 11^2 = 121$. By the "Pascal's Triangle" trick, $121^2 = 14641$, and the desired sum is 16.

6		C	Find the number of degrees in the acute angle formed by the hour and minute hands of a standard analog clock at 2:15 PM. A) $30°$ B) $7.5°$ C) $22.5°$ D) $5°$ E) Answer not given. Per the formula $\frac{\lvert 60h-11m\rvert}{2}$ and letting $h=2$ and $m=15$, the answer is $22.5°$.
	6	B	Suppose x varies directly with y^3, and y^5 varies directly with z. Then x^5 varies directly with z^n. Find n. A) 4 B) 3 C) 2 D) 1 E) Answer not given. $$x = ky^3$$ $y^5 = jz \to y = Jz^{1/5}$, for some constant J. Plugging this into the first equation gives us $$x = k\left(J^3 z^{\frac{3}{5}}\right) = Cz^{3/5}$$ for some constant C. Which means $x^5 = C^5 z^3$, thus making $n = 3$.
7	**7**	C	If x, $2x+2$, $3x+3$, … form a geometric sequence, what is the fourth term? A) 27 B) 13.5 C) -13.5 D) -27 E) Answer not given. Calculate the common ratio in two ways and set them equal: $$\frac{2x+2}{x} = \frac{3x+3}{2x+2}$$ This yields solutions $x \in \{-1, -4\}$. Turns out that $x = -1$ doesn't make sense since that makes the second-term onward to be 0. Thus, $x = -4$, making the common ratio to be $3/2$, and the fourth term is $(3x+3)\left(\frac{3}{2}\right) = -13.5$

8	**8**	E	Find the positive difference of the roots in the equation $x^2 - 7x = 9$. A) -9 B) 3.5 C) -3.5 D) 9 E) Answer not given. If a and b are the roots, then by standard math team formulae, $a + b = 7$ and $ab = -9$. Also, notice that $$(a-b)^2 = (a+b)^2 - 4ab$$ Thus, the square of the difference is $(-7)^2 - 4(-9) = 85$, so the positive difference is $\sqrt{85}$.
9		C	Which of the following is the smallest positive number? A) $3\sqrt{11} - 10$ B) $45 - 2\sqrt{506}$ C) $51 - 10\sqrt{26}$ D) $18 - 5\sqrt{13}$ E) Answer not given. We can rewrite the expressions as A) $\sqrt{99} - \sqrt{100}$ B) $\sqrt{2025} - \sqrt{2024}$ C) $\sqrt{2601} - \sqrt{2600}$ D) $\sqrt{324} - \sqrt{325}$ We want the smallest positive number, so A and D are out. Since the function $f(x) = \sqrt{x} - \sqrt{x-1}$ is a decreasing function, the bigger the x, the smaller the expression. Thus, the answer is C.
	9	B	What is the sum of the positive integral divisors of 792? A) 2350 B) 2340 C) 2330 D) 2320 E) Answer not given. Since $792 = 2^3 3^2 11$, per the standard formula, the sum of the factors is $$(1+2+4+8)(1+3+9)(1+11) = 2340$$

10	**10**	A	For real number x, let $M(x)$ equal the minimum value of $x+2$, $-2x+4$, and $4x+1$. Find the largest possible value of $M(x)$. A) $8/3$ B) $5/2$ C) $2/3$ D) $1/2$ E) Answer not given. Interpret the expressions as lines. The graphs of $y = x+2$ and $y = 4x+1$ intersect at $\left(\frac{1}{3}, \frac{7}{3}\right)$, and the graphs of $y = x+2$ and $y = -2x+4$ intersect at $\left(\frac{2}{3}, \frac{8}{3}\right)$. Also, by graphing, we can see which line is below another, and we find that: $M(x) = \begin{cases} 4x+1 & x < 1/3 \\ x+2 & 1/3 < x < 2/3 \\ -2x+4 & x > 2/3 \end{cases}$ Which has a global maximum of $8/3$.

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High School

Team Test Solutions

9/ 10th	11/ 12th	Answer	Solution
1		5	Suppose $\cot(195°) = a + \sqrt{b}$, where a and b are positive integers and b is not divisible by a perfect square. Find $a + b$. $\cot(x) = \frac{\cos(x)}{\sin(x)}$ $\cot(195°) = 2 + \sqrt{3}$
2		128	A sphere with a volume of 2 cubic meters is similar to another sphere whose radius is four times that of the original sphere. Find the number of cubic meters in the volume of the larger sphere. $2 \times 4^3 = 128$
3		-945	What is the coefficient of the x^4 term in the expansion of $(x-3)^7$, with like-terms combined? $7C4 * (-3)^3$

4	**1**	16000	Train A is set to leave station X heading east at a speed of 4 meters per second. Train B is set to leave station Y heading west at a speed of 6 meters per second. A bird, traveling at 10 meters per second, leaves station X heading east at the same time train A leaves. When the bird reaches train B, train B will depart. The bird also turns around and heads back toward train A. The bird then continuously goes back and forth between the two trains. Given that station X and Y are 10,000 meters apart, how many meters will the bird have traveled when the trains meet? The bird first flies 10,000 meters. Then flies 10 m/s for another 600 seconds (the time train B travels before meeting train A)
5	**2**	14	Urn #1 has 167 red and 143 blue balls. Urn #2 has 4 red balls and 4 blue balls. One ball is picked at random from Urn #1 and placed in Urn #2. Then, two balls are picked at random from Urn #2. Let a/b equal the probably that the two balls are different colors, where a and b are relatively prime positive integers. Find a + b. Regardless of what color is picked first, Urn #2 will have 5 of one color and 4 of another. The probability of this $\frac{C(5,1)C(4,1)}{C(9,2)} = \frac{5}{9}$. Averaged across all possible ball choices in Urn #1, we get 5/9.
6	**3**	67	The infinite series $$\sum_{x=1}^{\infty} \frac{9}{(3x-2)(3x+7)}$$ is equal to a/b, where a and b are relatively prime positive integers. Find $a+b$. Telescoping series, because $$\frac{9}{(3x-2)(3x+7)} = \frac{1}{3x-2} - \frac{1}{3x+7}$$ Answer is $39/28$.

7	**4**	40	Let ax + by + cz = D, be an equation of the plane spanned by vectors <4, -6, -1> and <-6, 0, 5>, where a, b, and c are relatively prime positive integers. Find a + b + c. The cross-product vector of the two spanning vectors is parallel to <a, b, c>.
8	**5**	27	Lines are drawn from a point, A, outside a circle with center C and radius $\sqrt{3}$ to the two tangent points, B and D, on the circle. Segment AB has length 3. Let T be the area inside triangle ABD but outside of the circle. Find $(T+\pi)^2$. Let C be the center of the circle. Find area of kite ABCD minus area of the sector BDC. Area is $T = 3\sqrt{3} - \pi$.
9	**6**	181	Let c be a value such that the below matrix does *not* have an inverse. Find the value of 11 c. $$\begin{pmatrix} 11 & 0 & 4 \\ 13 & -3 & c \\ -3 & 1 & -5 \end{pmatrix}$$ This occurs when the determinant is zero. $$c = \frac{181}{11}$$
10	**7**	7	Suppose that $\log_{10} a = \log_4 b = \log_{25}(4a+b)$. Let $\frac{a}{b} = M + \sqrt{N}$, where M and N are positive integers and N is not divisible by a perfect square. Find $M+N$. $$10^x = a, 4^x = b, 25^x = 4a+b$$ Then $$\frac{a^2}{b^2} = \frac{4a+b}{b}$$ Then solve $$\left(\frac{a}{b}\right)^2 - 4\left(\frac{a}{b}\right) - 1 = 0$$ And a/b must be positive. The desired solution is $\frac{a}{b} = 2 + \sqrt{5}$.

	8	8	Betty is attempting to cross a river in a canoe. The river is $16 + 4\sqrt{2}$ meters across. The current is moving at a speed of 4 meters per second toward a waterfall, which is $16 + 4\sqrt{2}$ meters downstream. Assuming Betty can row at a speed of 3 meters per second, what is the fastest time she can get across the river while avoiding falling down the waterfall? Express your answer in seconds. Let x be the horizontal row speed, and y be the vertical row speed. Then betty's movement is (4 – x) m/s down river. And y m/s across the river. Then $$x^2 + y^2 = 3^2$$ and $$4 - x = y$$
	9	3	Find the sum of all positive primes p such that $$p^{2020} + p^{2021}$$ is a perfect square. $$p^{2020}(1+p) = k^2$$ This implies $$1 + p \text{ is a perfect square}$$ Thus $$1 + p = m^2$$ $$p = (m-1)(m+1)$$ Where m is an integer This is only possible when m=2 thus p=3
	10	0	Let Q(x) be the quotient when $$45x^{74} - 74x^{45} + 29$$ is divided by $$x - 1.$$ Compute the sum of the coefficients of Q(x), including the constant coefficient term. The expression equals $$45(x^{74} - 1) - 74(x^{45} - 1)$$ When divided by x-1 is $$45(x^{73} + \cdots + 1)$$ $$-74(x^{44} + \cdots + 1)$$ Then Q(1) = $$45(74) - 74(45) = 0$$

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High School

Pressure Round Solutions

9/ 10th	11/ 12th	Answer	Solution
1		14	Given triangle with sides 7 and 5 with angle x between them. If cos(x) = 3/5, find the area of the triangle. Area of Triangle = (1/2)*a*b*sin(x). Cos(x) = 3/5 implies sin(x) = 4/5
2	**1**	23	If Joe can paint a house in 3 hours and Sam can paint the same house in 5 hours. Let a/b be the number of hours it takes for them to paint the house together where a and be are relatively prime positive integers. Compute a + b. Joe paints 1/3 of the house per hour Same paints 1/5 of the house per hour Their combined rate of painting is $$\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$$ Thus they will finish in 15/8 hours.
3		4	Find the units digit of 19!/15!. Expressions is the same as $$19 * 18 * 17 * 16$$ Using mod10 arithmetic =9*8*7*6=2*7*6=4*6=4mod10
4	**2**	3	Find the sum of the roots of the polynomial: $$3x^3 - 9x^2 - x = -3$$ -b/a 9/3=3
5	**3**	2	Evaluate: $$\cos^2(18°) + \cos^2(36°) + \cos^2(54°) + \cos^2(72°)$$ Use identities of $$\cos(x) = \sin(90 - x)$$ And $$\cos^2(x) + \sin^2(x) = 1$$

	4	288	On her vacation, Sarah brought 2 hats, 3 scarves, 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many unique outfits can she make? Assume a hat and a scarf are optional items. (2+1)*(3+1)*4*3*2
	5	792	How many ways are there to distribute $55 one-dollar bills to six people such that all six get at least $8? Give everyone 8 first. Then give out the remaining 7. A 12 digit sequence of x's and |'s will determine who gets how much. $e.g\ x\|xx\|\|xx\|x\|x$ The number of rearrangements of this is $\frac{12!}{7!\,5!}$

"Math is Cool" Championships -- 2020-21

High School

College Bowl Round #1 Solutions

	Answer	Solution
1	103	What is the sum of the largest and smallest positive prime factors of 2020? Prime factors are 2, 5, 101.
2	111[base 2]	Express as a binary (base 2) number: $6\log_4 2 + \log_4 256$. Only the digits of your answer need to be provided, or in other words, do not include the base. Basic log rules.
3	22	It costs $1.75 to wash a load of laundry and $1.50 to dry a load of laundry. If you are washing and drying 3 loads of laundry and additionally only washing 7 loads of laundry, how much money did you spend? Express your answer in dollars. Do not include a decimal point and cents. 10 loads total are washing and 3 loads are drying.
4	3,780	How many ways can you rearrange the letters in the word HASHTAGS so that the As are not next to each other? There are $\frac{8!}{2!2!2!}$ ways to rearrange the letters in HASHTAGS in any order. There are $\frac{7!}{2!2!}$ ways to rearrange the letters in HASHTAGS so that the As are next to each other. The difference is the answer.
5	18	The area in the xy-plane bounded by the curve $$x^2 - 8x + 4y^2 + 8y - 16 = 0$$ is $k\pi$. What is k? Complete the square to get it into the form to find r_1 and r_2, which is $$\frac{(x-x_0)^2}{{r_1}^2} + \frac{(y-y_0)^2}{{r_2}^2} = 1$$
6	2	What is the product of the third and seventh digits after the decimal point in the decimal representation of 1/7? Repeating sequence 0.1428671…

7	2	Evaluate: $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ Set x = $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ So $x = \sqrt{2+x}$, and then solve for x. The positive value is the one that makes sense.
8	53	Let i equal the square root of -1. What is the product of $7 + 2i$ and its conjugate? (7 + 2i) * (7 – 2i)
9	75 [degrees]	What is the measure of the smaller angle between the hour and minute hands of a clock at 3:30? Express your answer in degrees. 30 degrees between each number on clock. Hour hand will be halfway between numbers 3 and 4 (15 degrees). Minute hand will be at 6.
10	403	The probability of drawing three consecutive cards with the same suit from a standard deck of 52 cards, without replacement, in its simplest form, is $^{n}/_{d}$. What is $d - n$? 1 * (12/51) * (11/50) = 22/425

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High School

College Bowl Round #2 Solutions

	Answer	Solution
1	-14,076	What is the 2020th term of an arithmetic sequence with first term 57 and common difference negative 7? 57 – 7(2019)
2	$2,700	When closing a deal on an apartment rental in NYC, real estate agents receive a commission that is 15% of the annual rent. If the monthly rent is $1,500, the agent receives how many dollars? Do not include a decimal point and cents. Need to calculate annual rent ($1,500 * 12 = $18,000).
3	336	A solid box is 8 inches by 9 inches by 10 inches and is made of 1-inch cubes. If the outside of the box is painted, how many unit cubes are not painted at all? Volume of cube that is 2 inches smaller in each dimension to account for cubes that are painted on at least one side.
4	6	How many positive integer divisors of 6,075 are perfect squares? 6075 = $3^5 * 5^2$ ^Find combination of perfect squares here 1 is also a perfect square
5	4 [units]	The area of a trapezoid is 144 units squared. $B_1\ is \frac{1}{6}\ h\ and\ B_2\ is \frac{1}{3} h$. What is B_1? $$A = \frac{B_1 + B_2}{2} h$$
6	218	The determinant of matrix A below is equal to d. What is the sum of the positive factors of d (not including d)? $$A = \begin{bmatrix} 9 & 1 \\ 11 & 19 \end{bmatrix}$$ 9 * 19 – 1 * 11 = 160 Factors are 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80

7	13	There are 5 unfair coins, where the probability of flipping heads is 60%. The probability that exactly 2 of them show heads when these 5 coins are flipped is $^{n}/_{d}$, as a reduced fraction. Find the value of $\sqrt{d} - \sqrt{n}$. $$\binom{5}{2}(.6)^2(.4)^3 = \frac{144}{625}$$
8	2,187	What is twice the sum of the terms in the following infinite geometric series? 729, 243, … 2* (729 / (1 – 1/3))
9	9001	Vegeta wants to know what is the first prime number that is over 9000. What is that number? You just need to know that this is a prime number.
10	-64	Calculate the sum of coefficients in the expansion of $(3x - 7)^3$. Plug in 1 to find the sum.

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College Bowl Round #3 Solutions

	Answer	Solution
1	0807	What is the date of the 220th day of the year 2020? State as mmdd (ie: January 1 = 0101). Include leading zeroes in your answer. Start each month at 30 days 7*30 = 210 Calculate +1 -1 +1 +0+1+0+1 for up to end of July = 213 **2020 is a leap year (29 days)!
2	504	What is the least common multiple of 24 and 63? 24 = $2^3 * 3$ 63 = $3^2 * 7$
3	12	The area of a 60° angle sector of a circle is 6π. The circumference of the circle kπ. What is k? 60° is 1/6 of circle. Area of circle is 36π. Radius is 6, circumference is 12π.
4	7	Richard is trying to record a 15 second *TikTok* video. It takes him 3 seconds to “floss,” 5 seconds to “do the robot,” and 2 seconds to “disco.” How many combinations of dance moves can Richard use in his video? (He can repeat dance moves) Robot 3x Robot 2x, floss 1x, disco 1x Robot 1x, disco 5x Robot 1x, floss 2x, disco 2x Floss 5x Floss 3x, disco 2x, Floss 1x, disco 6x

5	8	Three tennis balls, with diameter 2 inches each, are stacked perfectly on top of each other in a cylindrical can that has a diameter of 2 inches and 12 inches in height. The volume of the space inside the can but outside the tennis balls is $k\pi$ cubic inches. What is k? Calculate the volume of the 3 tennis balls and subtract from the volume of the cylinder. 3 tennis balls = 4*pi*(1)^3, Cylinder = 12*pi*1^2
6	28	Given eight points in the plane, find the maximum number of lines that can be drawn by connecting any of the two points. Trick: $\binom{8}{2}$
7	2	The point (x, y) is 2/3 the distance between (18,2) and (-3, -10), and is closer to (-3, -10). What is the absolute value of the sum of the coordinates of (x, y)? 2/3 distance is (4,-6) Sum is -2 Absolute value is 2
8	3	How many integers between 20 and 50, inclusive, have an odd number of positive integral factors? Only square numbers have an odd number of factors
9	60	What is the sum of the 5^{th} and 10^{th} terms in the "Fibonacci" sequence? Assume the first two Fibonacci numbers are both equal to 1. 1,1,2,3,**5**,8,13,21,34,**55** **5+55**
10	58	You see people, some with dogs, on a sunny day. You count 158 legs and 57 heads. 80% of people are wearing sunglasses. How many eyes (individual, not pairs of eyes) do you see? Assume that no dogs are wearing sunglasses. 2p+4d = 158 legs 35 people, 22 dogs 7 without sunglasses 29 pairs of eyes, 58 individual eyes