"Math is Cool" Masters -- 2021-22

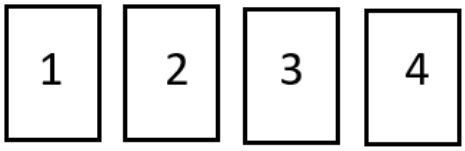
High School

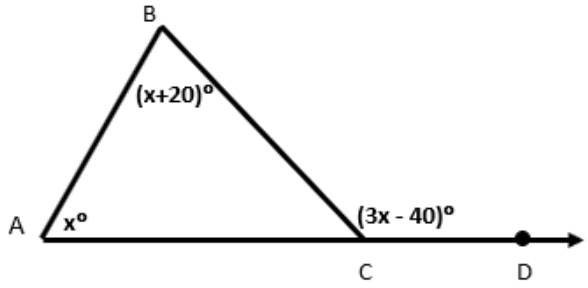
Mental Math Solutions

	Answer	Solution
1	3	The number 19 can be expressed as the sum of three distinct prime numbers. What is the smallest of the three prime numbers? **3+5+11 = 19**
2	50 [°]	How many degrees are in the acute angle formed by the hands of a clock at 2:20 pm? **Every 5 minutes on the clock = 30°. The minute hand is at 20 minutes, so from 15 – 20 minutes = 30°. The hour hand is 1/3 of the way in-between 2 and 3, so it is 20° away. Total = 50°.**
3	2	How many of the numbers in the following set are greater than the mean of the set? {1, 2, 3, 4, 5, 12} **Mean = 27/6 = 4.5**
4	4000	Gregg buys four new tires and a new spare tire for his tractor. He rotates the tires, including the spare tire, so that after driving 5000 miles, every tire has been used for the same number of miles. For how many miles was each tire used? **5000 miles * 4 active tires = 20000 tire-miles, divided by 5 total tires = 4000 miles each**
5	203 [inches]	The perimeter of a rectangle is 2006 inches. Its length is 800 inches. What is the width of the rectangle in inches? **2006 – 2(800) = 406** **406/2 = 203**
6	55	What is the smallest composite number generated by $n^2 - n - 1$, where n is a positive integer? **$8^2 - 8 - 1 = 55$**
7	5	How many arithmetic sequences have at least 3 positive integer terms with a first term equal to 3 and a last term equal to 21. **Possible values of d, the common difference, are: 1, 2, 3, 6, or 9.**
8	21	What is the value of x if $4^{10} + 4^{10} = 2^x$? **$(2^2)^{10} + (2^2)^{10} = 2^x$** **$2^{20} + 2^{20} = 2^x$** **$2(2^{20}) = 2^x$** **$2^{21} = 2^x$**

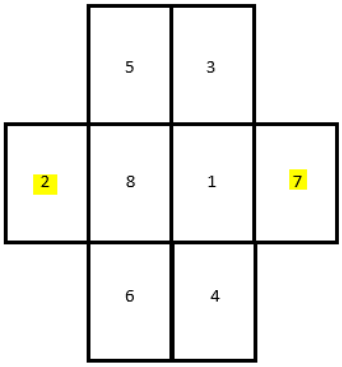
"Math is Cool" Masters -- 2020-21
High School
Individual Test Solutions

	Answer	Solution
1	2	How many prime factors does 12 have? **2 and 3.**
2	14	Let x = $5^{-2} + 10^{-1}$. What is 100x? $\frac{1}{25} + \frac{1}{10} = \frac{14}{100}$
3	37	In a division problem, the divisor is 11, the quotient is 3, and the remainder is 4. What is the dividend? **37/11 = 3r4**
4	16	The value of $\sqrt{260}$ is between two whole numbers. What is the larger of the two numbers? **16^2 = 256, 17^2 = 289**
5	-1002	A line segment on the coordinate plane has endpoints at (1000, -3000) and (-3004, 2004). The midpoint of the segment has coordinates (x, y). What is x? x: $\frac{1000 - 3004}{2} = -1002$
6	18 [miles]	If town A and town B are eight miles apart, and town C is ten miles from town B, what is the furthest possible distance (in miles) from town A to town C? **Furthest case possible is that C is 18 miles away.** 

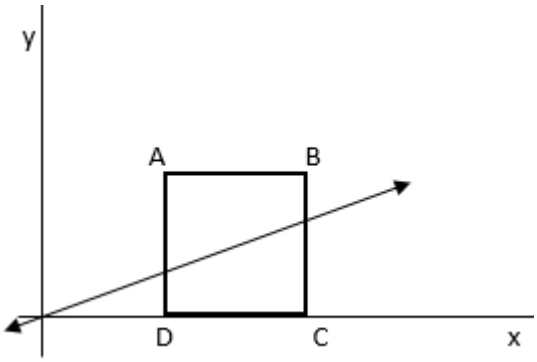
7	3	You are shown the following four cards. Each card has a single positive integer 1, 2, 3, or 4 on each side, but you can see only one side of each card. Numbers may be repeated on both sides of a card. How many cards must you turn over to verify that any card that has a 2 on one side also has a 4 on the opposite side?  **Need to turn over the '1' and '3' to verify that they don't have a '2' on the other side. Need to turn over the '2' to verify that it has a '4' on the other side. Don't need to turn over the '4' because it doesn't matter.**
8	10	Solve for N: 6!·7! = N! **7! = 7x6x5x4x3x2x1** **6! = 6x5x4x3x2x1 = 2x3x5x4x3x2** **Beyond the 7!, the 6! adds a factor of 8 (2x4), a factor of 9 (3x3) and a factor of 10 (5x2)**
9	54 [$]	If 6 gallons of premium unleaded gasoline costs $21.60, how many dollars would it cost to completely fill a 15-gallon tank? $\frac{\mathbf{6}}{\mathbf{21.6}} = \frac{\mathbf{15}}{\boldsymbol{x}}$ **x = 15(21.6)/6 = 54**
10	25	When a single card is drawn from a standard 52-card deck, the probability that it is not a king can be written as a reduced common fraction A/B. What is A + B ? $\frac{\mathbf{48}}{\mathbf{52}} = \frac{\mathbf{12}}{\mathbf{13}}$

11	-6	A line parallel to the line given by 2x + 3y = 6 passes through the point (x, 2), and intersects the y-axis at the point (0, -2). What is the value of x in the given point (x, 2)? **2x + 3y = 6 can be written as:** **y = (-2/3)x + 2** **The parallel line has the same slope m = -2/3, with b = -2, and given y = 2** **y = mx + b** **2 = (-2/3)x – 2** **4 = (-2/3)x** **x = -6**
12	4 [elements]	Sets X, Y and Z are defined as follows: X = {a, c, e, g} Y = {a, b, c} Z = {b, c, d, e, f} How many elements are in the following set? $Y \cup (X \cap Z)$ $\boldsymbol{X \cap Z}$ **= {c, e}, intersection** $\boldsymbol{Y \cup (X \cap Z)}$ **= {a, b, c, e}, union**
13	768	A geometric sequence has a first term of 3, and a common ratio of 4. What is the 5th term in the sequence? **Multiply by 4 each time.** **3, 12, 48, 192, 768**
14	40 [degrees]	Triangle ABC has interior angles A and B as indicated, and exterior angle BCD as indicated. What is the measure of angle BCA, in degrees?  **Angle BCA = 180 – (3x – 40)** **Sum of three interior angles = 180.** **x + (x + 20) + 180 – (3x – 40) = 180** **x = 60** **Therefore, angle BCA =** **180 – (3·60 – 40) = 40 degrees**

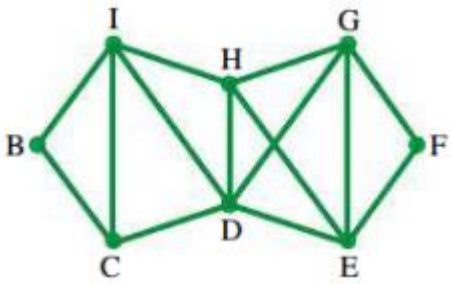
15	10	You have three alarms in your room. Your cell phone alarm is set to ring every 30 minutes, your computer alarm is set to ring every 20 minutes, and your clock alarm is set to ring every 45 minutes. If all three alarms go off simultaneously at 12:34 pm on Monday, the next time that they will all go off simultaneously the same day can be written as A:BC pm on an analog clock, where A, B and C are single digit positive integers. What is A + B + C ? **20 = 2^2x5** **30 = 2x3x5** **45 = 3^2x5** **LCM = 2^2x3^2x5 = 180 minutes = 3 hours** **Next time will be 3:34 pm.** **3+3+4 = 10**
16	15	Violet was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been if she had worked the problem correctly? **What Violet did:** **(x – 9)/3 = 43** **x = 138** **What she should have done:** **(138 – 3)/9 = 15**
17	27 [integers]	How many positive three-digit integers contain only the digits 1, 2, and 3, assuming that digits may be repeated? **There are three choices for each digit: 3x3x3 = 27 ways.**
18	14	One of the Platonic solids is a regular octahedron. What is the sum of the number of faces and the number of vertices of a regular octahedron? **8 faces and 6 vertices**

19	5	The digits 1, 2, 3, 4, 5, 6, 7, 8 are each to be placed in the following boxes, with one digit per box. No two consecutive numbers can be placed in boxes that are next to each other either horizontally, vertically or diagonally. For example, if the 5 is placed in the far left box, then neither the 4 nor the 6 can be placed in the box directly to the right of the 5 or in the two boxes that are diagonally above and below the 5. What is the positive difference between the two numbers that are placed in the left-most and the right-most boxes, labeled as 'L' and 'R'? **Here is one possible solution. The important thing to note is that 1 and 8 must always be in the middle two positions, and 2 and 7 must always be in the left-most and right-most positions. That frees up the other four boxes, since 1 and 8 do not have other consecutive partners**. 
20	3	The vertex of the graph $y = x^2 - 4x + 5$ is at the point (x, y). What is x + y? **The x-coordinate is x = -b/2a = 4/2(1) = 2. Plug that into the function to calculate $y = 2^2 - 8 + 5 = 1$.**

21	15 [paths]	In the following figure, how many different paths are there to spell the word MATH, by moving from the letter M to the letter A to the letter T to the letter H? Moves can be made in a vertical or horizontal direction only, no diagonal moves are allowed. **Notice that the figure is symmetric, so we only have to look at half of it. From the top middle M, there is one path (which does not have a symmetric partner). From the 2nd row left M, there are 3 paths. From the 3rd row left M, there are 3 paths. From the bottom row left M, there is one path. With symmetry, the total paths are: 1 + 2(3+3+1) = 15** 
22	5	In a sequence of numbers that begins 1, 3, 2, …, each term after the first two terms is defined to be equal to the term preceding it minus the term preceding that. Therefore, the 3rd term as shown above = 3 – 1 = 2. Find the sum of the first 100 terms of the sequence. **Generating the first 6 terms of the sequence results in the following: 1, 3, 2, -1, -3, -2. Notice that the sum = 0. Starting with the 7th term, this pattern of six terms repeats itself. 96 is a multiple of 6, so the sum of the 1st 96 terms is 0. The sum of the next 4 terms is 5.**
23	17	An integer is randomly selected from 1 through 50. Given that the selected number is prime, the probability that it contains the digit 9 can be written as a reduced common fraction A/B. What is A + B? **There are 15 prime numbers between 1 and 50. Two of them contain the digit 9: 19, 29. The probability is 2/15.**

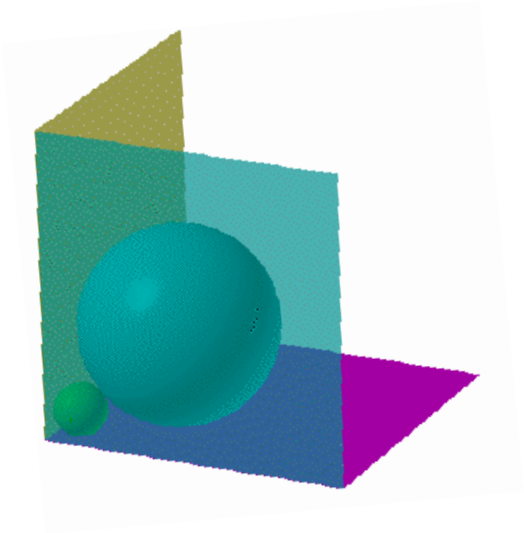
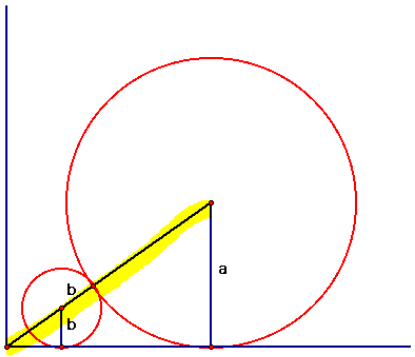
24	3	As shown in the figure, ABCD is a unit square with D located at (1, 0) and C at (2, 0). Find the reciprocal of the slope of the line through the origin that bisects the area of the square.  **The line has to go through the center of the square, which is point (3/2, ½). Therefore the slope m = 1/3, and the reciprocal is 3.**
25	7 [integers]	How many integers are not included in the solution to the following inequality? $$\left\lvert\frac{x+2}{3}\right\rvert > 1$$ $\boldsymbol{\frac{x+2}{3} > 1}$ ***or*** $\boldsymbol{\frac{x+2}{3} < -1}$ **Solving gives: x > 1 or x < -5** **The integers -5, -4, -3, -2, -1, 0 and 1 are not included.**
26	4 [units]	The combined volumes of two cubes with integer side lengths are numerically equal to the combined lengths of their edges. How many units long is the side length of the larger cube? **Let x = one side length, y = other side length** $\boldsymbol{x^3 + y^3 = 12x + 12y}$ $\boldsymbol{(x + y)(x^2 - xy + y^2) = 12(x + y)}$ $\boldsymbol{x^2 - xy + y^2 = 12}$ **Can do a little guess and check to discover that x = 2, y = 4 (or vice versa).**

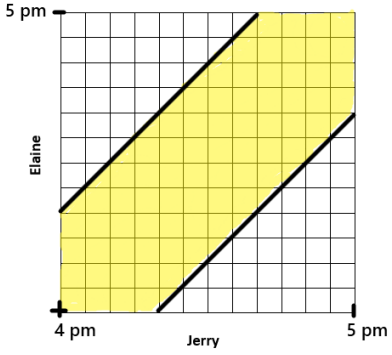
27	10000 [calculators]	The city of Calculusville has exactly 10,000 families. Each family owns exactly 0, 1 or 2 calculators. Every calculator belongs to a single family. More than half of the families have 1 calculator. Exactly half of the remaining families have 2 calculators. How many total calculators are there in Calculusville? **Reasoning: The majority have 1 calculator. Half of the rest have 2. If each of those families gave one of their calculators to the "other half of the rest", then every family would have exactly 1 calculator.**
28	15 [minutes]	If Vishal and William leave their houses at the same time, walking directly toward each other, each at their own constant rate, they will meet after five minutes. If Vishal leaves three minutes later than William, they meet after Vishal has walked for three minutes. How many minutes would it take Vishal to walk all the way from his house to William's house? **d = distance between their houses** **(i) $5(r_v + r_w) = d$** **(ii) $3r_v + 6r_w = d$** **Equate the two, and solve for:** **$2r_v = r_w$** **Substitute into (i):** **$5(3r_v) = d$** **$15r_v = d$** **Therefore, time = 15 minutes**
29	7	If f(n) is a function such that f(1) = f(2) = f(3) = 1, and such that: $$f(n) = \frac{f(n-1) \cdot f(n-2) + 1}{f(n-3)}$$ for any integer n > 3, then what is the value of f(6)? **f(4) = [(1)(1) + 1]/1 = 2** **f(5) = [(2)(1) + 1]/1 = 3** **f(6) = [(3)(2) +1]/1 = 7**
30	-6	Find the remainder when the polynomial $x^5 + x^4 + x - 5$ is divided by the binomial x + 1. $$\begin{array}{r} x^4+1 \\ x+1 \overline{)\, x^5+x^4+x-5} \\ \underline{-(x^5+x^4)} \\ x-5 \\ \underline{-(x+1)} \\ -6 \end{array}$$ **Or: f(-1) = (-1)5 + (-1)4 + (-1) – 5 = -6**

31	0	An Euler path is a path that goes through every edge of a graph exactly once, where an edge is the line between two vertices. An Euler circuit is an Euler path that begins and ends at the same vertex. For example, given square ABCD, draw a line from vertex A to B to C to D and back to A, which creates an Euler circuit. In the graph shown here, how many different vertices could be used as a starting (and ending) point to create an Euler circuit?  **A graph cannot have an Euler circuit if there are any odd vertices. In this graph, vertices C and D are odd, therefore there are 0 Euler circuits.**
32	16	There exist integer solutions (x, y) for the following equation. What is the largest sum x + y that corresponds to a particular solution to the equation? $(x - 4)(x - 10) = 2^y$ **The two solutions are (2, 4) and (12, 4), so the largest sum is 12 + 4 = 16.** **The quantities x – 4 and x – 10 must be powers of 2 that differ by 6, which implies 2 and 8, or -2 and -8.** **Alternatively, realize that the left hand side is a parabola and the right hand side is an exponential growth curve.** **Use a little guess-and-check to find the two solutions**.
33	280 [ways]	Mandelbrot Middle School has nine members on their math team. In how many ways can the nine students be divided into three teams of three students each? **$({}_9C_3 \cdot {}_6C_3 \cdot {}_3C_3)/3! = 1680/6 = 280$** **After choosing the groups of 3 students in the numerator, have to divide by 3! to account for the 6 different arrangements of the three groups.**

34	3 [integers]	Given that $i^2 = -1$, for how many integers n is $(n + i)^4$ an integer? **$(n + i)^4 =$** **$n^4 + 4n^3i + 6n^2i^2 + 4ni^3 + i^4$** **$= n^4 + 4n^3i – 6n^2 – 4ni + 1$** **$= n^4 + 4n(n^2 – 1)i – 6n^2 + 1$** **This will only be an integer if the imaginary term = 0. Therefore,** **$4n(n^2 – 1) = 0$** **$n = 0$ or $n^2 – 1 = 0$** **$n = 0, -1, 1$**
35	0	Let N = 1234567891011…998999 be the natural number formed by writing the integers 1, 2, 3, …, 999 in order. The left-most digit is '1', the second digit from the left is '2', and so on. What is the 2022^{th} digit from the left? **There are 9 single digits, followed by 90 2-digit numbers, for 189 digits. Every subsequent group of 100 numbers (100-199, 200-299, …) contains 300 digits. Therefore, the last digit in 699 is the 1989^{th} digit. Count up from there to see that the 2022^{th} digit is the '0' at the end of 710.**
36	60	Yesenia is running the last leg of a "fun run" relay race at her school's Sports Day. Her starting position is at the corner of the parking lot at coordinates (4, 4). She must run across the grass athletic field to touch the chain-link fence that has equation $2x + 2y + 6 = 0$. Then she must cross the finish line located at coordinates (-1, 7). The shortest distance that Yesenia can travel is written as a reduced radical $A\sqrt{B}$, where A and B are both integers. What is A + B? **Rewrite the equation of the fence as $y = -x – 3$. Reflect her starting location (4, 4) across the line to point (-7, -7). From this point, draw a straight line to the ending point of (-1, 7). Create a right triangle and solve for the hypotenuse, which is the distance Yesenia must travel. $14^2 + 6^2 = 232$. $\sqrt{232} = 2\sqrt{58}$.**

37	319	Ritika is stopped at a mile marker on an east-west county road. She decides to toss a fair coin 10 times. Each time, she will drive 1 mile east if the coin lands heads up, or she will drive 1 mile west if the coin lands tails up. After the 10 coin flips, the probability that Ritika ends up back at her starting point can be written as a reduced common fraction A/B. What is A + B? **In order to end up where she started, she must have flipped 5 heads and 5 tails, so she went east 5 miles, and west 5 miles. Using the binomial probability formula, P(5 heads) = $_{10}C_5 \cdot (1/2)^5 \cdot (1/2)^5$ = 252/1024 = 63/256**

38	5	Two spheres, a large exercise ball and a smaller plyometric ball, are placed in the corner of a storage closet, where the walls meet at a 90° angle. Both spheres are tangent to both walls, the floor, and are tangent to each other. The radius of the large exercise ball is 1 foot. The radius of the smaller plyometric ball, in feet, can be written in reduced radical form as $A - \sqrt{B}$, where A and B are positive integers. What is A + B?  **Let radius of the large exercise ball = 1, and radius of the smaller plyometric ball = b. Take a vertical slice through the centers of the two spheres, and at a 45 degree angle between the two walls.**  **The yellow-highlighted segment is the space diagonal of a cube with side length of a, therefore it equals $\sqrt{3} \cdot a$. It also equals a + b + $\sqrt{3} \cdot b$, using the same reasoning to find the distance from the center of the smaller circle to the corner. Set the two equal to each other:** **$\sqrt{3} \cdot a = a + b + \sqrt{3} \cdot b$** **Simplify to find the ratio of the two radii: $\frac{a}{b} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$** **For a = 1, solve for b = $2 - \sqrt{3}$.**

39	30 [integers]	How many four-digit positive integers that end in 75 are divisible by 75? **All numbers ending in 75 are divisible by 25. Therefore, for a 4-digit number ab75, we need (a+b) to be a multiple of 3. Examining the 4-digit numbers that end in 75: 1075, 1175, 1275, 1375, 1475, 1575, …** **1275 is the first one that meets the criteria, and adding 300 each time will bring us to another number ending in 75 that is divisible by 75. There are 90 total 4-digit numbers that end in 75, and 1/3 of them or 30 will be divisible by 75.**
40	14	Jerry and Elaine agree to meet at the coffee shop between 4 and 5 pm on Monday. They agree to each wait 20 minutes for the other to arrive, or to stay until 5 pm if their arrival time is after 4:40 pm. The probability that they will meet can be written as a reduced common fraction A/B. What is A + B? **The problem can be illustrated graphically as shown here. The yellow shaded region indicates the overlap of their possible times at the coffee shop. Probability = area (yellow) / total area = 80/144 = 5/9.** 

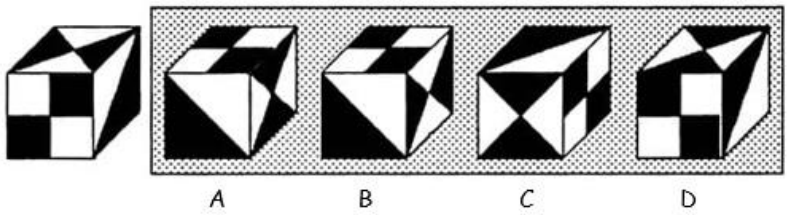
41	3	Matrix M shown here has an inverse matrix M^{-1}, which can be written: $$M^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ What is a + d ? $$M = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}$$ $$\left[\begin{array}{cc|cc} \mathbf{4} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{-6} & \mathbf{2} & \mathbf{0} & \mathbf{1} \end{array}\right]$$ $$\left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{-1/4} & \mathbf{1/4} & \mathbf{0} \\ \mathbf{-6} & \mathbf{2} & \mathbf{0} & \mathbf{1} \end{array}\right]$$ $$\left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{-1/4} & \mathbf{1/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{3/2} & \mathbf{1} \end{array}\right]$$ $$\left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{-1/4} & \mathbf{1/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{2} \end{array}\right]$$ $$\left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1/2} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{2} \end{array}\right]$$ **a + d = 1 + 2 = 3**
42	18 [correct answers]	A 25 question individual mathematics competition test is scored as follows: 5 points for a correct answer, -4 points for an incorrect answer, -3 points for no answer (left blank). Manuel scored a total of 64 points. How many correct answers did he submit? **Let C = # of correct answer, W = # of wrong answers, N = # of no answers.** **5C – 4W – 3N = 64** **C + W + N = 25** **You could combine the equations to eliminate one variable. Or, you could use a little guess-and-check to discover that C = 18, W = 5, N = 2.**

43	5	Find the sum of all possible solutions for 'x' in the following equation: log x – log 5 = log 2 – log(x – 3) **log(x/5) = log(2/(x-3))** **x/5 = 2/(x – 3)** **x(x – 3) = 10** **$x^2 – 3x – 10 = 0$** **(x – 5)(x + 2) = 0** **x = 5, x = -2** **x = -2 is not a solution because it is not in the domain.** **Therefore, the only possible solution is x = 5**
44	1	The following two conic sections intersect at a point (x, y) in Quadrant IV. What is x + y? $9x^2 – 4y^2 – 36x – 24y – 36 = 0$ $x^2 – 8x – y + 13 = 0$ **The first equation is a hyperbola:** $$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$$ **The center of the hyperbola is at (2, -3), and with a = 2, move right 2 units to point (4, -3)** **The second equation is a parabola:** **$y = (x – 4)^2 – 3$** **The vertex is at (4, -3), which is also on the hyperbola and in Quadrant IV.**
45	9	When solving the following equation for 'x' in radians, the largest solution for 'x' can be written as $\frac{A\pi}{B}$, where A and B are both integers. What is A + B ? $$sin^2(x) = \frac{1}{2}\sin(2x), 0 \leq x < 2\pi$$ $\boldsymbol{sin^2(x) = \frac{1}{2}(2\sin(x)\cos(x))}$ $\boldsymbol{sin^2(x) - \sin(x)\cos(x) = 0}$ $\boldsymbol{sin(x)(\sin(x) - \cos(x)) = 0}$ **sin x = 0 or sin(x) = cos(x)** **x = 0, π or sin(x)/cos(x) = 1** **tan(x) = 1** **x = π/4, 5π/4** **Largest solution is 5π/4**

"Math is Cool" Masters -- 2020-21

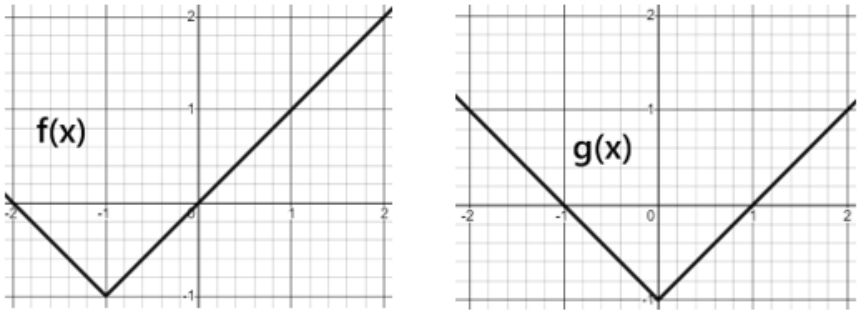
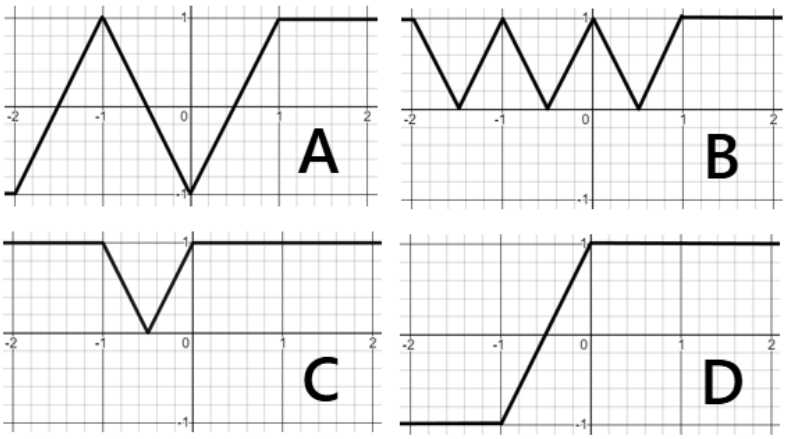
High School

Multiple Choice Solutions

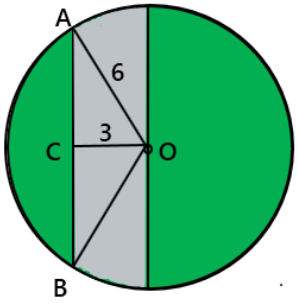
9/ 10th	11/ 12th	Answer	Solution
1	1	D	Which of the four cubes on the right could be another view of the three faces of the cube shown on the left?  A) Cube A only B) Cube B only C) Cube C only D) Cubes B and C E) Cubes C and D **Cubes B and C are rotations of the given cubes. Cubes A and D are not, which can be seen by looking at the boundary of the two faces containing the triangles.**
2	2	A	Perform the following addition: MCCCLIV + DXIII A) MDCCCLXVII B) MCDLXXII C) MCDLVII D) MMDLXXVII E) Answer not given. **1354 + 513 = 1867**

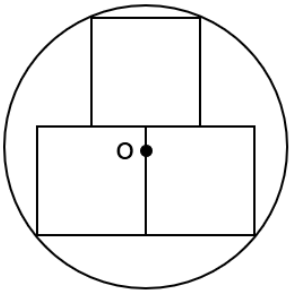
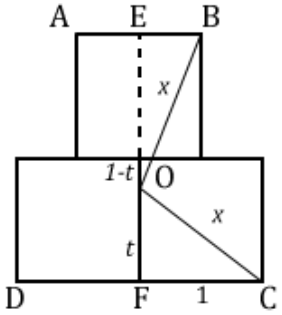
3		E	Given the following statements: 1. Some bips are not bops. 2. No bops are bleps. Assume that "some" means "at least one, but not all". Which of the following can be concluded? A) Some bips are not bleps. B) Some bleps are not bips. C) No bip is a blep. D) Some bips are bleps. E) Answer not given. **Can make a Venn Diagram, with an intersection between bips and bops. However, the relationship between the bips and bleps is completely unknown. The bleps could be completely separate from the bips, it could intersect with the bips, or it could be contained within the bips. Therefore, we cannot conclude any of these.**
	3	E	Given the following statement: If r > 0, then for all p and q such that pq does not equal 0, and such that pr > qr, Which of the following can be concluded? A) -p > -q B) -p > q C) $1 > \frac{q}{p}$ D) $1 < \frac{q}{p}$ E) Answer not given. **Since r > 0, can divide it out to conclude p > q. There are three cases to consider: p+q+, p+q-, and p-q-. In all cases, at least one example can be found to disprove each of the statements A – D.**

4	**4**	D	The mean age of a group of professors and administrators is 40 years. If the professors' mean age is 35 years, and the administrators' mean age is 50 years, what is the ratio of the number of professors to administrators? A) 3:2 B) 3:1 C) 2:3 D) 2:1 E) Answer not given. **x = no. of professors** **y = no. of administrators** **35x + 50y = 40x + 40y** **x/y = 2/1**
5	**5**	C	Three standard 6-sided dice are thrown. What is the probability that the sum of the numbers showing on the dice is 17? A) $\frac{1}{216}$ B) $\frac{1}{108}$ C) $\frac{1}{72}$ D) $\frac{1}{54}$ E) Answer not given. **There are three ways to get a sum of 17: 665, 656, 566.** **There are 6x6x6 = 216 total outcomes. P = 3/216 = 1/72**

6	6	C	The functions f(x) and g(x) are shown in the following graph.  Which of the following graphs could be the graph of the function: |f(x) – g(x)| ?  A) Graph A B) Graph B C) Graph C D) Graph D E) Answer not given. **Just pick points on f(x) and g(x), and calculate |f(x)-g(x)|. The correct graph is C.**
7		B	Given that x is a real number, and $x^{\frac{3}{2}} < 1$, which inequality describes the bounds of x? A) $0 \leq x \leq 1$ B) $0 \leq x < 1$ C) $x < 1$ D) $x > 1$ E) Answer not given. **Equivalent to:** $(\sqrt{x})^3 < 1$ **For x to be real, x must be ≥ 0. For the expression to be < 1, x must be < 1.**

	7	B	Assume that all of the following inverse trigonometric values are angles that can exist in a triangle, or are equal to 0. In radians, find the value of: $\sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$ A) π B) $\frac{3\pi}{4}$ C) $\frac{5\pi}{4}$ D) $\frac{\pi}{2}$ E) Answer not given. **$\sin^{-1}(1) = \frac{\pi}{2}$** **$\cos^{-1}(1) = 0$** **$\tan^{-1}(1) = \frac{\pi}{4}$** **Sum = $\frac{3\pi}{4}$**
8	8	C	If $f(x) = 4^x$, then $f(x + 1) - f(x) = ?$ A) 4 B) $2f(x)$ C) $3f(x)$ D) $4f(x)$ E) Answer not given. **$f(x + 1) - f(x) =$** **$4^{x+1} - 4^x =$** **$4^1 4^x - 4^x =$** **$4^x(4 - 1) =$** **$3 \cdot 4^x = 3f(x)$**
9	9	B	What is the sum of all possible products $x \cdot y$, where (x, y) is an integer solution to: $$xy - 2x + y = 7$$ A) 0 B) 12 C) 22 D) 24 E) 28 **Subtract 2 from both sides:** **$xy - 2x + y - 2 = 5$** **$x(y - 2) + 1(y - 2) = 5$** **$(x + 1)(y - 2) = 5$** **Casework:** **$x = 4, y = 3$** **$x = 0, y = 7$** **$x = -6, y = 1$** **$x = -2, y = -3$** **12+0-6+6=12**

10		*C*	A circular grass plot 12 feet in diameter is cut by a straight gravel path that is 3 feet wide, one edge of which passes through the center of the circle. The number of square feet in the remaining grassy area is: A) $36\pi - 34$ B) $36\pi - 6\sqrt{3}$ C) $30\pi - 9\sqrt{3}$ D) $35\pi - \frac{9\sqrt{3}}{2}$ E) Answer not given.  The right green half of the circle is 18π square feet. The left-hand green portion is equal to the area of sector AOB minus the area of triangle AOB. Sides AO and CO are known, which indicates that triangles AOC and BOC are 30-60-90 triangles. The missing sides (AC and BC) are therefore $3\sqrt{3}$. Angle AOB = 120°, so the area of sector AOB = 12π. The combined area of the two triangles is $9\sqrt{3}$. Total green area = $18\pi + 12\pi - 9\sqrt{3} = 30\pi - 9\sqrt{3}$

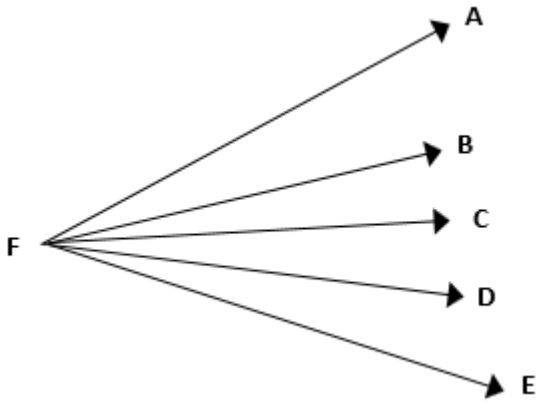
	10	D	See the following figure, which shows three unit squares (side length equal to 1) inscribed in circle O. The top square is positioned exactly in the middle of the bottom two squares, such that the midpoint of the top square's bottom edge lies on the point where the bottom squares' upper corners meet. What is the radius of the circle?  A) $\sqrt{2}$ B) $\frac{\sqrt{5}}{2}$ C) $\frac{5\sqrt{2}}{14}$ D) $\frac{5\sqrt{17}}{16}$ E) Answer not given.  **The radius is x, which is the hypotenuse of two right triangles: OEB and OFC. From OFC:** **$x^2 = t^2 + 1^2$** **From OEB:** **$x^2 = (2-t)^2 + (1/2)^2$** **Set the two expressions for x^2 equal to each other, and solve for t = 13/16.** **Therefore:** **$x^2 = (13/16)^2 + 1$** **Simplify to x = $\frac{5\sqrt{17}}{16}$**

"Math is Cool" Masters -- 2020-21

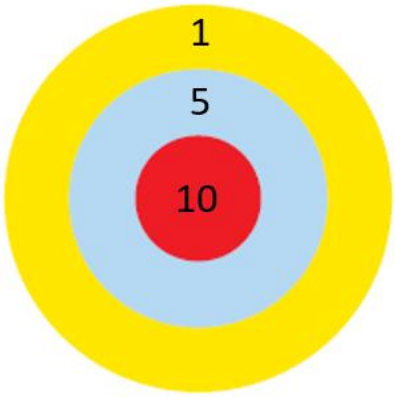
High School

Team Test Solutions

9/ 10th	11/ 12th	Answer	Solution
1	**1**	**23 [years]**	Jinu is five years old. His mother is 28 years old. How many years old will Jinu be when he is exactly half as old as his mother? **x = years until Jinu is half as old as his mom** **$5 + x = ½(28 + x)$** **$5 + x = 14 + x/2$** **$½\ x = 9$** **$x = 18$**
2	**2**	**[A=] 8**	For what base value A will 3201_A equal the base 10 number 1665? **The base must be 4 or higher. Use some "educated" guessing and check to find that: $3201_8 = 3x8^3 + 2x8^2 + 0x8^1 + 1x8^0 = 1536 + 128 + 0 + 1 = 1665$.**
3		**8**	Find the 2022nd digit after the decimal point in the decimal representation of 4/7. **4/7 = $0.\overline{571428}$** **It repeats every 6 digits. 2022/6 = 337 exactly. Therefore, the last digit will be 8.**
	3	**1872**	A three-digit number ending in 1 is given as ab1, where 'a' and 'b' represent the first two digits respectively. The sum of the digits, a + b + 1, is a two-digit number given as cd. The product of the digits c and d equals 8. What is the sum of all possible values for the original number ab1? **Both 'a' and 'b' must be ≤ 9, so the maximum value of cd is 9+9+1 = 19, and the minimum value is 10. Therefore, cd must = 18, to get a product of 8. Therefore, a and b must be 9 and 8, or 8 and 9. 981 + 891 = 1872.**

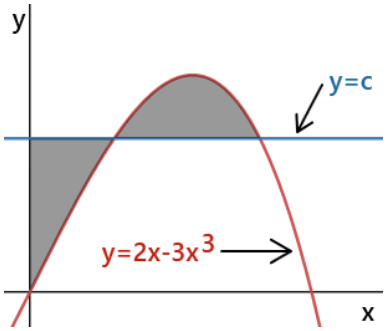
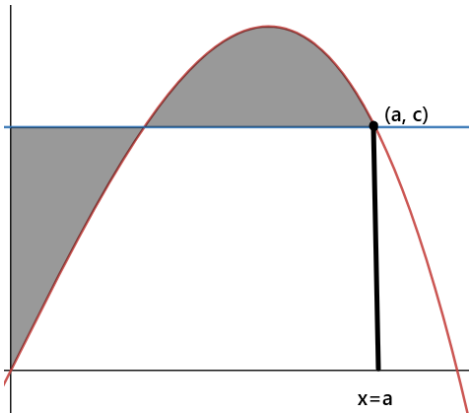
4	**4**	**6 [times]**	Nishka rode her bike $\frac{3}{4}$ of the way home before getting a flat tire. She walked the rest of the way home. If her walking time was twice as much as her biking time, how many times faster is her biking speed than her walking speed? **Walking distance = d, biking distance = 3d.** **Biking time = t, walking time = 2t.** **Bike rate: r_b = 3d/t** **Walk rate: r_w = d/2t** **r_b/r_w = (3d/t)/(d/2t) = 6**
5		**9 [acute angles]**	Angle AFE is an acute angle. How many other acute angles are shown in the diagram, not including ∠AFE?  **The following angles are also acute:** **AFB, BFC, CFC, DFE** **AFC, BFD, CFE** **AFD, BFE**
	5	**279 [integers]**	How many four-digit positive integers contain the digit pattern '75' at least once? The digits are located immediately next to each other as indicated. **Three cases:** **1. 75 _ _ , 10*10 = 100 numbers** **2. _ 75 _, 9*10 = 90 numbers** **3. _ _ 75 , 9*10 = 90 numbers** **However, we have counted 7575 twice, so subtract 1. Total = 279.**

6	**6**	**5 [androids]**	In a particular community, it is known that androids always lie, and humans always tell the truth. Collectively, androids and humans are referred to as "beings". In a group of 10 "beings", the following statements are made: • The first being says: "At least one of us is an android". • The second being says: "At least two of us are androids". • The third being says: "At least three of us are androids". • … The pattern continues until the tenth being says: "At least ten of us are androids". How many of the "beings" are androids? **Suppose there are 6 or more androids. Then at least 6 of the statements are true, which would mean they were made by humans, which contradicts that there are 6 or more androids. Suppose there are 4 or fewer androids. Then at least 6 statements are false, but that contradicts that there are 4 or fewer androids. Therefore, there must be 5 androids. The first through the 5^{th} are humans telling the truth, "at least 1 … 5 of us is an android". The 6^{th} through the 10^{th} are androids lying, "at least 6 … 10 of us are androids".**
7	**7**	**365**	Determine the next number in the sequence: 5, 15, 37, 77, 141, 235, … **Use the method of successive differences. Listing the differences between each pair of terms gives: 10, 22, 40, 64, 94. List the differences between each pair of differences: 12, 18, 24, 30. The difference between each of these values is 6. Work backwards to obtain the next term which is 365.**

8	**8**	**[A + B =] 13**	The dart board shown below consists of three concentric circles. The smallest circle (red) has a radius of 1 unit, the middle circle (blue) has a radius of 2 units, and the outer circle (yellow) has a radius of 3 units. The points shown are awarded for landing on each part of the dartboard, for example if your dart lands in the red section, you get 10 points. If you throw a dart, assume that it will land in a random location somewhere on the board. The expected value of this game, in points, can be written as a reduced fraction A/B. What is A + B ?  **Calculate the area of each region:** **inner (red): $A = \pi 1^2 = \pi$** **middle (blue):** **$A = \pi 2^2 - \pi = 4\pi - \pi = 3\pi$** **outer (yellow):** **$A = \pi 3^2 - 4\pi = 9\pi - 4\pi$** **$= 5\pi$** **Total area = 9π** **x P(x) xP(x)** **10 1/9 10/9** **5 3/9 15/9** **1 5/9 5/9** **Sum of xP(x) = 10/9 + 15/9 + 5/9 = 30/9 = 10/3**

9	**9**	**[A+B+C=] 14**	The points A (1, 2), B (0, 0), and C (-1, 3) are plotted on the coordinate plane, forming angle ABC. The line y = mx + b is the angle bisector of ∠ABC. The slope of the line, m, can be written in the form: $A + B\sqrt{C}$, where A, B and C are integers. What is A + B + C ? **Use the given points to write the equations of the lines containing AB and CB: y = 2x, and y = -3x. In standard form:** **2x – 1y + 0 = 0** **3x + 1y + 0 = 0** **Use the distance formula to equate the distances from each line to a point on the angle bisector:** $$\left\lvert\frac{3x+y}{\sqrt{10}}\right\rvert = \left\lvert\frac{2x-y}{\sqrt{5}}\right\rvert$$ **Removing the absolute value bars to solve will lead to two solutions – by inspection of a graph, the one with the steep upward slope is desired. The correct equation to solve is:** $$\frac{3x+y}{\sqrt{10}} = \frac{-(2x-y)}{\sqrt{5}}$$ **Rearrange the first equation into y = mx+b form and rationalize the denominator to get:** **m = 7 + 5$\sqrt{2}$**

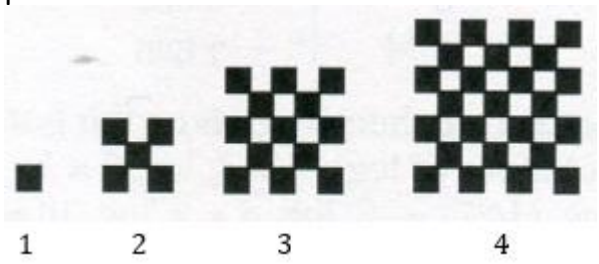
10		**15**	What is the sum of all the distinct values of x that satisfy the following equation? $$(x^2 - 5x + 5)^{x^2-9x+20} = 1$$ **Consider three cases:** **1) the exponent = 0** **$x^2 - 9x + 20 = 0$** **$(x - 5)(x - 4) = 0$** **$x = 4, 5$** **2) the base = 1** **$x^2 - 5x + 5 = 1$** **$x^2 - 5x + 4 = 0$** **$(x - 4)(x - 1) = 0$** **$x = 1, 4$** **3) the base = -1 and the exponent is even** **$x^2 - 5x + 5 = -1$** **$x^2 - 5x + 6 = 0$** **$(x - 2)(x - 3) = 0$** **$x = 2, 3$, both values result in the exponent being even.** **Therefore,** **$x = \{1, 2, 3, 4, 5\}$**

	10	**[A+B=] 13**	A horizontal line y = c intersects the curve y = 2x – $3x^3$ in the first quadrant of the coordinate plane, as shown in the figure. Find the value of c as a reduced fraction A/B, such that the areas of the two shaded regions of the graph are equal. What is A + B ?  Drop a perpendicular line from the far point of intersection to the x-axis, designated as x = a. Therefore the point of intersection is (a, c), where c = 2a – $3a^3$. The area under the function y = 2x – $3x^3$ from 0 to a is equal to the area under the horizontal line y = c from 0 to a. $$\int_0^a (2x - 3x^3)dx = ca$$ $$a^2 - \frac{3}{4}a^4 = (2a - 3a^2)a$$ Simplify and solve for: $$a = \frac{2}{3}$$ Substitute and solve for c: $$c = 2 \cdot \frac{2}{3} - 3 \cdot \left(\frac{2}{3}\right)^3 = \frac{4}{9}$$ 

"Math is Cool" Masters -- 2020-21

High School

Pressure Round Solutions

9/ 10th	11/ 12th	Answer	Solution
1	**1**	761 [black tiles]	The first four figures of a pattern built with black tiles is shown following. How many black tiles will be required to build the twentieth figure in this pattern?  1 2 3 4 **There are various ways you could figure this out. One way is to notice that each iteration requires $n^2 + (n-1)^2$ tiles, where n = the iteration number. This is seen visually by the full rows containing 'n' tiles, and the shorter rows containing 'n – 1' tiles. Therefore, $20^2 + 19^2 = 761$**
2	**2**	6066 [gadgets]	The ratio of widgets to gadgets is 2 to 3, and there are 2022 more gadgets than widgets. How many gadgets are there? **w:g is 2:3, w/g = 2/3, 3w = 2g** **g = w + 2022** **g = (2/3)g + 2022** **(1/3)g = 2022** **g = 6066**
3		26	A set of integers {1, 6, 9, 12} has a 5th integer 'n' added to it, which is not equal to any of the other four integers. The new set of 5 integers has its median equal to its mean. What is the sum of all possible values of 'n' ? **The possible values of the median are 6, 9 or 'n'. Case work for each of the three possibilities:** **1) if median = 6, 'n' = 2** **2) if median = 9, 'n' = 17** **3) if median = 'n', n must be either 7 or 8, and a value of n = 7 works.** **2+7+17 = 26**

	3	36 [fewer solutions]	Rashida is writing a homework problem for her Algebra students. She wants them to find all solutions to the equation a + b + c = 12, where a, b, and c must all be non-negative integers, and {a, b, c} is an ordered triplet. She discovers that if she changes the "non-negative" condition of the problem for a, b and c to say "positive" instead, then the number of ordered triplet solutions will decrease. How many fewer solutions are there if a, b and c must all be positive integers? **The problem can be investigated by considering how many of the solutions to the original problem include either a, b or c equal to 0. Possibilities are:** **0, 0, 12 (3 ways)** **0, 1, 11 (6 ways)** **0, 2, 10 (6 ways)** **0, 3, 9 (6 ways)** **0, 4, 8 (6 ways)** **0, 5, 7 (6 ways)** **0, 6, 6 (3 ways)** **Therefore, 36 total ways to have a solution including 0.**
4	**4**	250074	The base-ten six-digit integer 2A00B4 is divisible by 9 and divisible by 11. What is the number? **The sum of the digits must be divisible by 9. 2+4 = 6, so A+B must equal 3 or 12. For divisibility by 11, take the alternating sum of digits from left to right, and that sum must be divisible by 11:** **2-A+0-0+B-4** **-A+B-2 must be divisible by 11** **Either A = 9 and B = 0, or -A+B-2=0. A = 9 and B = 0 does not work, so the equations are:** **-A + B = 2, and either** **1) A + B = 3, or** **2) A + B = 12** **Try both cases to find that A = 5, B = 7.**

5		50 [units squared]	Find the area in square units of Quadrilateral *ABCD* with the following vertices: *A* (-4, 3), B (2, 11), *C* (6, 8), D (0, 0) **Shoelace method:** **x y** **-4 3** **2 11** **6 8** **0 0** **-4 3** **A = ½|(-4)(11) + (2)(8) + (6)(0) + (0)(3) – [(3)(2) + (11)(6) + (8)(0) + (0)(-4)]|** **= ½ |-44 + 16 – (6 + 66)|** **= ½ |-28 – 72| = ½ (100) = 50** **Or, a sketch will reveal that it is a rectangle, so use the distance formula to find that L = 10 and W = 5.**
	5	126 [units squared]	Find the area in square units of Polygon *ABCDE* with the following vertices: *A* (-4, 3), B (1, 9), *C* (13, -6), D (5, -4), E (-2, -6) **Shoelace method:** **x y** **-4 3** **1 9** **13 -6** **5 -4** **-2 -6** **-4 3** **A = ½|(-4)(9) + (1)(-6) + (13)(-4) + (5)(-6) + (-2)(3) – [(3)(1) + (9)(13) + (-6)(5) + (-4)(-2) + (-6)(-4)]|** **= ½ |-36 – 6 – 52 – 30 – 6 – (3 + 117 – 30 + 8 + 24)|** **= ½ |-130 – 122| = ½ (252) = 126**

"Math is Cool" Masters -- 2020-21

High School

College Bowl Round #1 Solutions

	Answer	Solution
1	36	There are 50 unpaired socks in a sock drawer. Fifteen are pink, 15 are purple, and the rest are yellow. How many socks do you have to take out of the drawer to have guaranteed that at least one sock of each color was taken out? **You would have to pull all of the yellow socks, and then pink (or purple) socks. After that, then you would finally get one of the last color.**
2	2	Find the sum of the values of x that satisfy the following equation: $\log_4(x) + \log_4(x + 6) = 2$ **$\log_4[x(x+6)] = 2$** **$x(x + 6) = 16$** **$x^2 + 6x - 16 = 0$** **$(x + 8)(x - 2) = 0$** **$x = -8, 2$** **x = 2 is the only solution that is in the domain.**
3	876593	Construct the largest possible six-digit odd number, with no digit used more than once, and with a 9 in the tens place. **After placing the '9', start at the left with the largest remaining digits, 8765. The ones digit must be '3' to make it odd.**
4	16	Anita rolls 2 similar fair n-sided dice, where n > 5. What is n, if the most probable sum of the numbers obtained is 17? **n = 17-1** **Pattern is n+ 1 = most probable sum**
5	126 [ways]	How many ways can ten oranges be distributed among six people such that everyone receives at least one orange? **Stars and bars method: After distributing the first six oranges, there are 4 left to distribute, and it takes 5 bars to separate the six people. Therefore there are a total of 9 objects. No. of ways = 9!/(5!4!).**
6	4	What is the remainder when 3^{2022} is divided by 5? **The powers of 3 starting with 3^1 end in digits 3, 9, 7, 1, 3, 9, 7, 1, … The digits repeat in a pattern of 4. The last digit of 3^{2022} will be 9, so when dividing by 5 there will be a remainder of 4.**

7	103	A tennis ball is dropped from a height of twenty feet. Each time it hits the ground, it rebounds one-fourth the distance it has fallen. The total distance in feet that the ball will travel before it comes to rest can be written as a reduced common fraction A/B. What is A + B? **The down travels are a geometric series with $a_1 = 20$, and r = ¼. The sum is 80/3. The up travels are a geometric series with $a_1 = 5$ and r = ¼. The sum is 20/3. The total distance is the sum, which is 100/3.**
8	44	If $f(x) = 4x^2 + 17$, what is the value of f(6) – f(5)? **$f(6) = 4(6)^2 + 17 = 161$** **$f(5) = 4(5)^2 + 17 = 117$** **161 – 117 = 44**
9	26	A non-isosceles triangle has integral sides of 4, 5, and x. Find the sum of all possible values of x. **By the triangle inequality theorem, x must be in the set {2, 3, 4, 5, 6, 7, 8}. However, it cannot be 4 or 5, because it is non-isosceles. Therefore, the possible values for x are {2, 3, 6, 7, 8}.**
10	8	The following expression can be written as $A \times 10^B$, where A and B are single-digit integers. What is A + B? $$\frac{280 \times 10^6}{7 \times 10^3}$$ **Simplifies to 4×10^4**

"Math is Cool" Masters -- 2020-21

High School

College Bowl Round #2 Solutions

	Answer	Solution
1	8 [$]	April bought a dozen roses at six dollars each, and May bought half a dozen roses at twelve dollars each. What was the average price per rose, in dollars? **(12*6 + 6*12)/18 = 8**
2	21	Let P equal the product of 2,824,117,563 and 82,165,302,015. What is the number of digits in P? **Estimate by doing $2.8 \times 10^9 \times 8.2 \times 10^{10} = 2.3 \times 10^{20}$, which has 21 decimal digits.**
3	8	A teacher divided 100 pencils among a group of 12 boys and girls. The boys each got 7 pencils, and the girls each got 9 pencils. How many girls were there? **b + g = 12** **7b + 9g = 100** **Solve for g = 8**
4	5	Two baseball teams are playing in a best-of-7 series. In other words, once one team wins 4 games, the series ends. The two teams are evenly matched, so each has the same chance of winning any given game. The probability that the series goes exactly 5 games can be written as a reduced common fraction A/B. What is A + B? **Negative binomial distribution: x = 5 games, r = 4 wins, p = 0.5 probability of winning each game. $b(5) = 4C3(0.5)^4(0.5)^1 = 0.125$ for each team. Therefore, 0.125(2) = 0.25 = ¼.**
5	864	The number of cubic feet in the volume of a cube is the same as the number of square inches in its surface area. What is the length of the edge of the cube in feet? **Let s = edge length in feet** **$V = s^3$** **$SA (in^3) = (12s)(12s)(6) = 864s^2$** **$s^3 = 864s^2$** **s = 864**

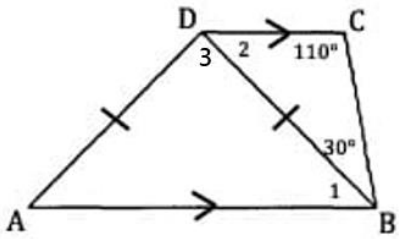
6	131	The mean of 11 numbers is 121. When one number is removed from the set, the average of the remaining 10 numbers is 120. What number was removed from the set? **The sum of the original 11 numbers is 11(121) = 1331. The sum of the remaining 10 numbers is (10)(120) = 1200. Therefore, 1331 – 1200 = 131 was removed.**
7	9	Suppose that you have an infinite supply of 4-cent and 7-cent stamps. How many postage amounts between 1 cent and 1 dollar, inclusive, cannot be made using these stamps? **The Frobenius number of 4 and 7 = (4)(7)-4-7 = 17, which is the maximum value that cannot be attained. All values that cannot be attained are: 1, 2, 3, 5, 6, 9, 10, 13 and 17.**
8	7	What is the largest prime factor of 504? **$504 = 2^3 3^2 7$**
9	-5	Find the sum of the values of x for which the following expression is undefined: $$\frac{2}{x(x^2+5x-14)}$$ **Fully factor the denominator:** **x(x + 7)(x – 2)** **x = 0, -7, 2**
10	30	Find the geometric mean of 12 and 75. $\sqrt{\mathbf{12 \cdot 75}} = \sqrt{\mathbf{3 \cdot 4 \cdot 3 \cdot 25}} = \mathbf{3 \cdot 2 \cdot 5}$

"Math is Cool" Masters -- 2020-21

High School

College Bowl Round #3 Solutions

	Answer	Solution
1	43	The following expression can be written as a reduced common fraction A/B. What is A + B? $\frac{1}{10}-\frac{1}{15}+\frac{1}{20}-\frac{1}{30}+\frac{1}{40}$ **= 9/120 = 3/40**
2	7	The mean of the following seven numbers is 4. What is the median of the seven numbers? {x, 17, x + 4, 4x - 3, -16, 9, x - 4} **The sum of the numbers divided by seven equals 4. Solve for x = 3. Arrange the numbers in order, the middle value = 7.**
3	9	Suppose that Amtrak has train service from Chicago IL to Detroit MI, and also from Detroit to Chicago, with trains leaving every hour on the hour from each city. The trip from one city to the other takes 4.5 hours, and all trains travel at the same speed. If you are on the train from Chicago to Detroit, how many trains going the other way will you pass? **Suppose you left Chicago at noon. You would pass the trains that had left Detroit at 8 am (due in at 12:30 pm), 9 am, 10 am and 11 am. Additionally, by the time you arrive at 4:30 pm, you would pass the trains that left at noon, 1, 2, 3, and 4 pm.**
4	699	Evaluate the following expression: $\left(\begin{vmatrix}5 & 6\\ 2 & -3\end{vmatrix}\right)^2-\sum_{n=1}^{4} n^2$ **$[5(-3)-(6)(2)]^2 - [1 + 4 + 9 + 16]$** **$[-15-12]^2 - 30$** **$[-27]^2 - 30$** **729 – 30 = 699**
5	-24	Find the sum of the values of x that satisfy the following equation: |x + 12| = 1794 **x = -1806, 1782**

6	3	A box contains fewer than twenty marbles. If you reach into the box and randomly pull out two marbles without replacing them, you have a 50% chance of getting two blue marbles. How many blue marbles were in the box to begin with? **There are 3 blue marbles and 1 non-blue. P(b&b) = (3/4)(2/3) = ½**
7	148	Evaluate, and give the answer in base 10 (do not include the base 10 in the answer): $111001_2 + 1011011_2$ **57 + 91 = 148**
8	100 [°]	In trapezoid ABCD shown here, AB is parallel to CD, and BD = AD. Angle DCB = 110° and angle CBD = 30°. What is the measure of angle ADB in degrees?  **∠2 = 40° (triangle sum = 180)** **Therefore, ∠1= 40° (opposite angles)** **Therefore, $\angle A$= 40° (isosceles triangle)** **Therefore,, $\angle ADB$= 100°**
9	97	Simplify the following expression to a reduced common fraction A/B. What is A + B? $$\left(\frac{64}{729}\right)^{-\frac{2}{3}}$$ **Simplifies to 81/16**
10	11	The decimal number 0.375 can be expressed as a reduced common fraction A/B. What is A + B ? **1/8 = 0.125, 3/8 = 0.375**