"Math is Cool" Championships -- 2021-22

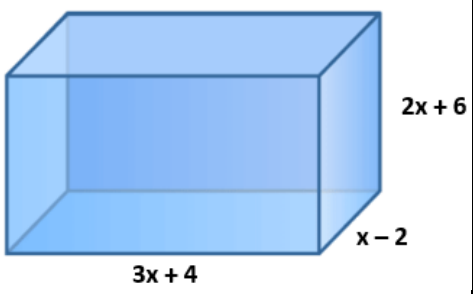
High School

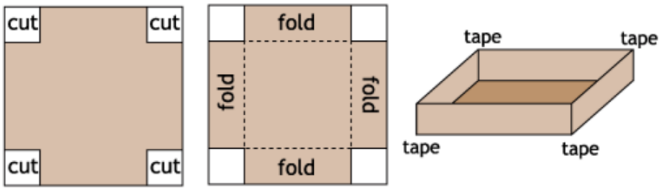
Mental Math Solutions

	Answer	Solution
1	165	What is the sum of the odd integers between 4 and 26? **Sum of 1, 3, 5, …, 25 = 13^2 = 169, subtract off 4 = 165.**
2	16	What is 12 times the slope of the line with equation: 4x – 3y = –6 ? **Slope = 4/3**
3	5	If two fair six-sided dice are rolled, in how many ways can a sum of six be obtained? **1/5, 2/4, 3/3, 4/2, 5/1**
4	9	If f of x equals one-third x minus eight, what is f-inverse of -5? **f(9) = -5, therefore $f^{-1}(-5)$ = 9**
5	111 (base 4)	What is the sum of 32 base 4 and 13 base 4. Answer as a base 4 number, but do not include the base 4 in your response. **Do the addition in base 4.**
6	6	What is the area of an ellipse with a major axis of twelve over pi and a minor axis of 2? **A = πab, where a = semi-minor axis and b = semi-major axis. Therefore, A = (π)(6/π)(1)**
7	-5	What is the sum of the roots of the following equation: $x^3 + 5x^2 – 26x – 120 = 0$ **Sum of the roots of a polynomial = -b/a = -5/1 = -5**
8	52	I am dealt three cards from a standard 52-card deck. In how many ways can I get a three of a kind, that is, three cards of the same value. For example, three fives, or three Jacks, etc. **There are 13 different denominations (2 through ace). For each denomination there are 4 cards, so 4C3 = 4 ways. 13x4 = 52.**

"Math is Cool" Championships -- 2020-21
High School
Individual Test Solutions

	Answer	Solution
1	15	An arithmetic sequence has a first term of 3 and a second term of 7. What is the value of the 4th term of the sequence? **3, 7, 11, 15**
2	16	Evaluate: $64^{\frac{2}{3}}$ $\left(\sqrt[3]{\mathbf{64}}\right)^{\mathbf{2}} = \mathbf{4^2 = 16}$
3	[m=] -8	A line has the equation $y = \frac{1}{8}x + 6$. What is the slope of a line that is perpendicular to it? $\boldsymbol{y = mx + b}$, **perpendicular slope is** $\mathbf{-\frac{1}{m} = -\frac{1}{\frac{1}{8}} = -8}$.
4	48[°]	What is the measure of the angle in degrees that is complementary to the angle 42°? **90-42 = 48**
5	15 [cm]	A right triangle has a leg length of 8 cm and a hypotenuse of 17 cm. What is the length of the other leg in cm? $\mathbf{\sqrt{17^2 - 8^2} = \sqrt{289 - 64} = \sqrt{225} = 15}$
6	[x=] -1	Solve the equation for 'x'. 5(3x + 2) = -5 $\mathbf{5(3x + 2) = -5 = 15x + 10}$; $\mathbf{-15 = 15x; x = -1}$
7	96	When $(x^3y^2)^4$ is simplified to x^ay^b, what is the product ab? $\mathbf{(x^3y^2)^4 = x^{12}y^8}$. $\mathbf{12(8) = 96}$.

8	[a=] 3	The lines given by the following equations intersect at the point (1, 2). $3x + 2y = 7$ $4x - ay = -2$ What is the value of 'a' ? **Substitute (1,2) into the second equation:** $\mathbf{4(1) - a(2) = -2}$ $\mathbf{6 = 2a. a = 3}$
9	21	What is 15 percent of 140? $\mathbf{\frac{15}{100}(140) = \frac{3}{20}(140) = 3(7) = 21.}$
10	[f(-2)=] -33	Find the value of $f(-2)$ when $f(x) = -3x^2 + 5x - 11.$ $\mathbf{f(-2) = -3(-2)^2 + 5(-2) - 11}$ $\mathbf{= -12 - 10 - 11 = -33}$
11	[x=] -7	The slope of a line passing through the points (x, 5) and (-2, 10) is 1. What is the value of 'x' ? $\mathbf{\frac{5-10}{x-(-2)} = 1, -5 = x + 2, x = -7}$
12	[n=] 108	If the sequence 12, 36, n, …. is geometric, what is the value of 'n' ? **36/12 = 3, 3*36 = 108**
13	56	If Ax + B is the sum of the twelve edges of the rectangular prism shown here, what is the value of A+B?  $\mathbf{4[(3x + 4) + (x - 2) + 2x + 6)]}$ $\mathbf{= 4[6x + 8] = 24x + 32.}$ **24+32=56**
14	[x=] 3	Given $f(x) = 5x + 7$, find the value of 'x' so that $f(x) = 22.$ **5x + 7 = 22, 5x = 15, x = 3**

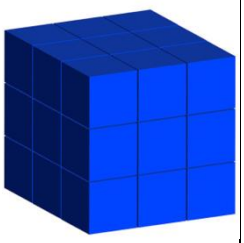
15	4	What is the average of the values of x that satisfy the equation $x^2 - 8x - 20 = 0$? $\boldsymbol{x^2 - 8x - 20 = 0}$ $\boldsymbol{= (x - 10)(x + 2)}$ $\boldsymbol{\frac{10 + (-2)}{2} = 4\ or}$ $\boldsymbol{\frac{sum}{2} = -\frac{b}{2} = \frac{8}{2}}$
16	2[mph]	Franny walked 1/8 of a mile in 1/16 of an hour. What is Franny's walking rate in miles per hour? $\boldsymbol{r = \frac{d}{t} = \frac{\left(\frac{1}{8}\right)}{\frac{1}{16}} = 2\ mph}$
17	13500 [cu in]	ASB members are placing boxes around the school to collect canned goods for a food drive. To make the boxes, they start with a square sheet of cardboard with a side length of 60 inches. They cut out congruent 15 by 15 inch squares from each corner of the cardboard sheet, then fold the sides up and tape the edges together as shown. What will be the volume of each box?  $\boldsymbol{(60 - 30)(60 - 30)15}$ $\boldsymbol{= 30(30)(15) = 13500\ cu\ in}$

18	[A + B =] 8	Four blue gummy worms and three yellow gummy worms are placed in a bag and selected one at a time at random. The gummy worms are selected without replacement. The probability that three gummy worms of the same color were selected without replacement can be written as the reduced fraction A/B. What is A + B? **Find P(BBB) + P(YYY)** **P(BBB) =** $\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}=\frac{4}{35}$ **P(YYY) =** $\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}=\frac{1}{35}$ **P(BBB or YYY)** $=\frac{5}{35}=\frac{1}{7}$ **A+B = 8**
19	47	What is the largest prime factor of 2021? **Testing some factors are easy (2, 3, 5, 11,…), some take a while (7, 13, …). Note though that 7(11)(13) = 1001, so 2002 is divisible by 7, 11 and 13, which are therefore not factors of 2021.** **However:** $\mathbf{2021=2025-4=45^2-2^2=}$ $\mathbf{(45-2)(45+2)=43(47)}$.
20	[x=] -7	What is the x-coordinate of the image of the point (–5, 7) after it is rotated 90 degrees counterclockwise around the origin? **Rotating 90 degrees interchanges the x and y coordinates. (-5,7) is in quadrant 2, rotating counter-clockwise will be in quadrant 3 so both x and y are negative.** **(-7,-5)**
21	6480 [sq in]	How many square inches are in 5 square yards? **5 square yards = 5(36)(36) = 6480 square inches**
22	28	The number 1001 is raised to the 6th power. What is the sum of the digits of the result? $\mathbf{(1001)^6=(1000+1)^6}$. **Using the binomial expansion, 1 to any power is 1 and 1000 times itself just adds 3 zeroes. So the result is just the binomial coefficients (6th row of Pascals triangle) in block of 3 digits. 1,006,015,020,015,006,001. Digits sum to 28.**
23	128	What is the value of the largest power of 2 that evenly divides into $19^4 - 13^4$? $\mathbf{19^4-13^4=(19^2-13^2)(19^2+13^2)}$ $\mathbf{=(19-13)(19+13)(19^2+13^2)}$ $\mathbf{=6(32)(530)=6(32)(2)(265)}$; $\mathbf{2(32)(2)=128}$

24	72	Find the sum of the 3 smallest positive integers 'c' such that $\sqrt[3]{108c}$ is also an integer. **Need all exponents of prime factors to be multiples of 3.** $$\mathbf{108c = 2^2 3^3 c}$$ $$\mathbf{c = 2^1 = 2, 2^4 = 16, 2^1 3^3 = 54}$$ $$\mathbf{2 + 16 + 54 = 72}$$
25	10 [values]	A 4-digit positive integer is divisible by 9. How many possible values are there for this integer, given that the last two digits of the number are 25? **Need the sum of the digits to be divisible by 9, ab25 means:** $$\mathbf{a + b = 2, a + b = 11.}$$ **ab=11, 20, 29, 38, 47, 56, 65, 74, 83, 92. 10 values total.**
26	12 [points]	How many points on a circle centered at the origin with radius 13 have both x and y coordinates that are integers? **By symmetry, one only needs to check one quadrant with one half-axis. In quadrant 1, (13,0),(12,5),(5,12); 3 points. 3(4)=12**
27	5 [seconds]	Biff and Eho climbed to the top of a cliff 720 feet high and threw a ball straight down at a speed of 64 ft/sec. Use 32 ft/sec^2 as the gravitational constant. How many seconds will it take for the ball to hit the ground? **Equation of motion:** $$\mathbf{h(t) = \frac{1}{2}at^2 + v_0 t + h_0}$$ $$\mathbf{= -16t^2 - 64t + 720}$$ **Set equal to height 0 for the ground and solve for t.** $$\mathbf{t^2 + 4t - 45 = (t + 9)(t - 5) = 0}$$ **5 seconds.**

28	[A + B =] 5	At the end of a treasure hunt, Parker discovers two treasure chests. One of them contains 100 gold coins, and the other contains 50 gold coins and 50 silver coins. Parker opens a chest at random, draws a coin from it at random, and it is gold. The probability that he opened the chest with 100 gold coins can be written as a reduced fraction A/B. What is A + B ? **Let *G* be the event that a gold coin is drawn; *F* the event that the box with 50 gold is drawn and *H* the box with 100 gold.** $$P(H|G) = \frac{P(H \text{ and } G)}{P(G)}$$ $$= \frac{P(H \text{ and } G)}{P(G|H)P(H) + P(G|F)P(F)}$$ $$= \frac{\left(\frac{1}{2}\right)}{\left[(1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]}$$ $$= \frac{2}{2+1} = \frac{2}{3}. \quad 2+3=5.$$
29	42	I roll a fair six-sided die. What is the expected number of rolls until I get 2 sixes in a row? **Let *e* be the expected number of rolls and conditioned on the outcome of the first few rolls. If the first roll is NOT a six (prob=5/6), you start over and thus the expected total rolls is (*e*+1). If the first roll is a 6 but the second is not (prob=(1/6)(5/6)), then you start over and expected total rolls is (*e*+2). If the first 2 rolls are both 6 (prob=1/36), you are done and it took 2 rolls. The equation is then:** $$e = \left(\frac{5}{6}\right)(e+1) + \left(\frac{5}{36}\right)(e+2) + \left(\frac{1}{36}\right)(2)$$ **Solving, *e*=42.**
30	6 [sq units]	Tom found the area of a regular hexagon with a perimeter of 12 units to be $x\sqrt{3}$ square units. What is the value of x? **Break the hexagon into 6 equilateral triangles of side 2.** $$A = 6\frac{s^2\sqrt{3}}{4} = 6\sqrt{3}.$$

31	8 [positive integers]	How many positive integers less than 1000 have exactly 9 positive integer factors? **Number of factors of:** $$n = p_1^{e_1}p_2^{e_2}\ is\ (e_1+1)(e_2+1)$$ $$n = 2^8, 2^2 3^2, 2^2 5^2, 2^2 7^2,$$ $$2^2 11^2, 2^2 13^2, 3^2 5^2, 3^2 7^2$$ **8 values total.**
32	8	Let f(x) be defined for all real numbers and f(x – 4)f(x + 4) = 72 for all x, and f(x) = x + 12 on (-4, 4). What is f(2021)? $(x-4)f(x+4) = 72$ **so** $$f(x+8) = \frac{72}{f(x+16)}$$ $$f(x) = \frac{72}{f(x+8)} = f(x+16)$$ **The function has a period of 16 so** $f(2021) = f(5) = \frac{72}{f(-3)} = \frac{72}{9} = 8$.
33	8	Let $a, b, c, x > 1$. Suppose $\log_{ab}(x) = 9, \log_{bc}(x) = 18, \log_{ac}(x) = 12$. What is $\log_{abc}(x)$? $$x = (ab)^9, ab = x^{\frac{1}{9}}.$$ $$x = (bc)^{18}, bc = x^{\frac{1}{18}}.$$ $$x = (ac)^{12}, ac = x^{\frac{1}{12}}.$$ $$(abc)^2 = x^{\frac{1}{9}+\frac{1}{18}+\frac{1}{12}} = x^{\frac{1}{4}}$$ $$abc = x^{\frac{1}{8}}, \log_{abc}(x) = 8$$
34	144 [subsets]	How many (possibly empty) subsets of the following set of integers have no consecutive integers? {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} **Call the subsets w/o consecutive integers "good" sets. The number is R_{10}. Any subset in R_9 will be good in R_{10} and not include '10'. Include each subset in R_8 and include '10'. So $R_{10} = R_9 + R_8$. Starting the sequence is easy.** $$R_0 = 1, R_1 = 2, R_2 = 3, R_3 = 5$$ $$R_4 = 8, R_5 = 13, R_6 = 21,$$ $$R_7 = 34, R_8 = 55, R_9 = 89$$ $$R_{10} = 144$$

35	27 [schools]	At the state math contest, each school brought 3 students to compete. As it turns out, each student had a different score. The student that had the median score was the highest from his school while his teammates were 44th and 78th respectively. How many schools attended the contest? **44th place was below the median so there were less than 44+43=87 students or 29 schools competing. A student was 78th place so at least 78/3 = 26 schools were there. There was a unique median so there was an odd number of students and thus schools. There must have been 27 schools.**
36	7 [solutions]	On the interval from $[0, \pi]$, how many solutions are there to the following equation? cos(3x) = cos(7x) $$\cos(7x) - \cos(3x) = \cos(5x + 2x)\cos(5x - 2x) = 0$$ $$= 2\sin(5x)\sin(2x).$$ $$\sin(2x) = 0, x = 0, \frac{\pi}{2}, \pi$$ $$\sin(5x) = 0, x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi$$ **A total of 7 solutions.**
37	[A + B =] 10	A 3x3x3 cube is painted blue on the outside, while the inside surfaces remain white. The cube is then cut into 27 1x1x1 cubes. One of the 27 cubes is chosen at random, and rolled like a 6-sided die. If the face that shows on the top is blue, the probability that it is the only blue face on that cube can be written as a reduced fraction A/B. What is A + B?  **There are 27 small cubes with a total of 162 faces. As selected each face is equally likely to be on top; but only 54 are blue so the others are eliminated. Of those, only 6 are on cubes where that is the only Blue face. 6/54 = 1/9. 1+9=10**
38	[n=] 18	The polynomial $x^3 - 6x^2 + 3x + n$ has three roots: r, s, and t. If r = s + t, what is the value of n? **The sum of the roots is $-(-6) = r + s + t = 2r; r = 3$. Now $3 = rs + rt + st = r(s + t) + st = 9 + st; st = -6$. So,** $$n = -rst = -(3)(-6) = 18.$$

39	10	The following fifteen integers are distributed to three friends, each getting five: 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 5 After selecting their numbers, the friends make the following statements: Alice: The product of my five numbers is not divisible by 9 or 10. Benny: The product of my five numbers is not divisible by 5 or 6. Carli: The product of my five numbers is not divisible by 3 or 4. What is the sum of Carli's five numbers? **Alice can have at most one 3 and cannot have the 5 if she has a two or four which would mean her other values would be 1's. All even.** **Benny can't have the 5, if he has a 3, he can't have 2 or 4. All odd.** **Carli has the 5 as it places no condition on the others. She cannot have a 3 or 4 and at most one 2.** **Alice: 4, 4, 2, 2, 2** **Benny: 3, 3, 3, 1, 1** **Carli: 5, 2, 1, 1, 1** **Total of Carli's is 10.**
40	29	Savi is trying find all possible 4th vertices (a, b) of a parallelogram with vertices (2, 2), (3, 5) and (7, 10). What is the sum of all possible a's and b's ? **Ex. Connect (2,2) and (3,5) over 1 up 3. Do the same with (7,10) to get (8,13). There are 3 points. (6,7) (8, 13) (-2, -3)** **6+7+8+13+-2+-3 = 29**

"Math is Cool" Championships -- 2020-21

High School

Multiple Choice Solutions

9/ 10th	11/ 12th	Answer	Solution
1	1	D	A point P with coordinates (x, y) is reflected across the x-axis to point P'. Which of the following is the rule for this transformation? A) $P(x,y) \rightarrow P'(x,y)$ B) $P(x,y) \rightarrow P'(-x,y)$ C) $P(x,y) \rightarrow P'(-x,-y)$ D) $P(x,y) \rightarrow P'(x,-y)$ E) Answer not given. **Reflecting across the x-axis is a vertical move. The value of x remains the same, the value of y changes to its opposite.**
2	2	C	A carpenter has a board 72 inches long that must be cut into 10 pieces of equal length. Each cut causes a waste of $\frac{3}{16}$ of an inch of board. How many total inches of board will remain after the cuts have been made? A) $70\frac{1}{4}$ B) $70\frac{5}{8}$ C) $70\frac{5}{16}$ D) $70\frac{1}{8}$ E) Answer not given. **There will be a total of 9 cuts, therefore need to subtract 3/16 a total of 9 times.** **72 – 9(3/16) = 1125/16 = 70 5/16**
3	3	B	Put the following numbers in order from smallest to largest: 42_7 , 37_8 , 11101_2 A) $42_7 < 37_8 < 11101_2$ B) $11101_2 < 42_7 < 37_8$ C) $42_7 < 11101_2 < 37_8$ D) $37_8 < 42_7 < 11101_2$ E) Answer not given. **Convert each number to base 10:** **42_7 = 30** **37_8 = 31** **11101_2 = 29**

<table>
<tr>
<td>4</td>
<td>4</td>
<td>A</td>
<td>Pavarti has the following information about her box of "mixed chocolate" candies.

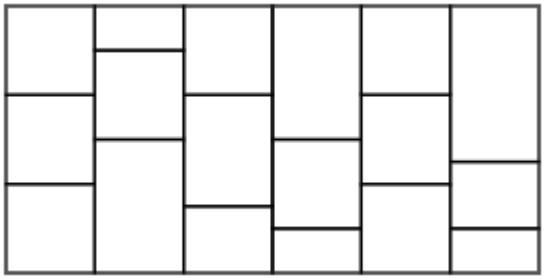
There are 24 total candies.
9 of them contain both nuts and caramel.
8 of them contain neither nuts nor caramel.
13 candies in total contain caramel.

How many of the candies contain nuts, but no caramel?

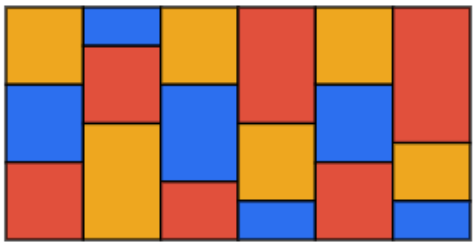
A) 3 B) 5 C) 6 D) 7 E) Answer not given.

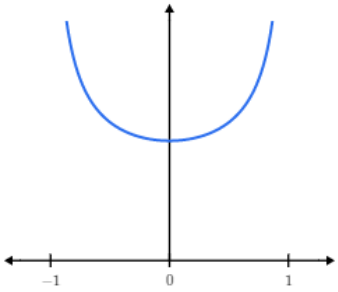
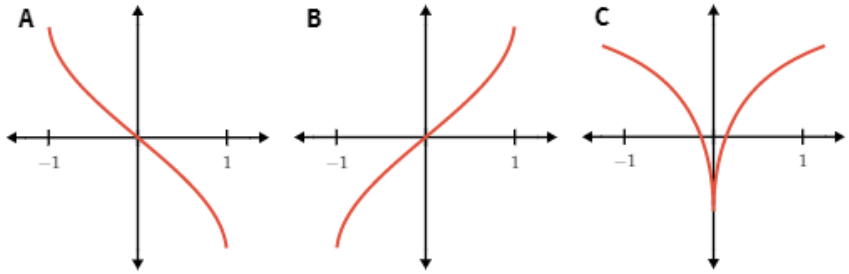
Construct a two-way table with the given information:

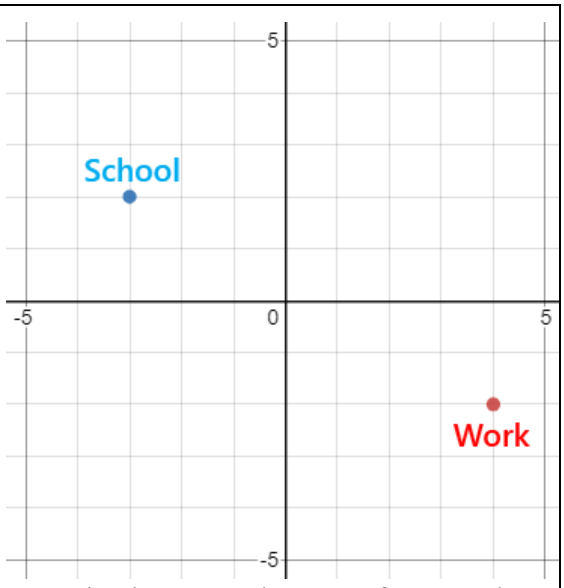
<table>
<tr><td></td><td>Contains nuts</td><td>Does not contain nuts</td></tr>
<tr><td>Contains caramel</td><td>9</td><td>4</td></tr>
<tr><td>Does not contain caramel</td><td>3</td><td>8</td></tr>
</table>
</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>A</td>
<td>What is the minimum number of colors required to fill the regions of this graph so that no two adjacent (sharing the same border) regions are the same color?


A) 3 B) 4 C) 5 D) 6 E) Answer not given.

The four-color theorem tells us that at most 4 colors will be needed. In this case, only 3 are needed.


</td>
</tr>
</table>

6		B	Two sides of an obtuse triangle are 12 and 21. Which of the following lengths can the third side not be equal to? A) 10 B) 20 C) 27 D) 32 E) Answer not given. **Sides 12-20-21 result in an acute scalene triangle. The other three options result in an obtuse triangle.**
	6	B	The following function may be the derivative of one of the functions shown below in graphs A, B or C. Which graph (if any) could it be the derivative of?   A) Graph A B) Graph B C) Graph C D) None of them. E) Answer not given. **Graph B is constantly increasing, so the derivative is always positive. It can't be Graph C because the derivative is not defined at x = 0, and it can't be Graph A because the function is constantly decreasing.**

7	7	E	In Calculusville, the streets follow a square grid, and all distances are measured in terms of vertical and horizontal units along these streets. For example, the distance from 'School' at (-3, 2) and the origin (0, 0) is 5 units, measured as 3 units over and 2 units down.  Kazuma wants to live at exactly the same distance from both his work located at (4, -2) and his school located at (-3, 2). How many potential grid points with integer coordinates can he consider? A) 1 B) 2 C) 4 D) 7 E) Answer not given. **Answer is 0. The vertical distance between school and work is 4, and the horizontal distance is 7, for a total distance of 11 units. Therefore, Kazuma would have to live 11/2 = 5.5 units from both school and work. However, this does not allow for any solutions with integer coordinates.**

8		B	In 2020, the population living in Washington State was approximately 7.7 million people. The state population increases about 1.3% every year. Assume that this population growth continues at the same rate. The population size in millions can be modeled by a function, P(t), which depends on the amount of time, t, in years, since the year 2020. Which of the following is the correct function P(t)? A) $P(t) = 7.7 \cdot e^t$ B) $P(t) = 7.7 \cdot (1.013)^t$ C) $P(t) = 7.7 \cdot (1.3)^t$ D) $P(t) = 7.7 \cdot (1.3)^{t/12}$ E) Answer not given. **Model the situation with an exponential function P(t) = $a \cdot b^x$, where a = initial quantity, b = factor by which the quantity is multiplied over constant time intervals, and x = expression in terms of t that determines the time intervals. In this case, a = 7.7 (million), b = 1.013 (representing 1.3% increase annually), and x = t (in years).**

	8	D	Anthony and his family are riding the Seattle Great Wheel, which is a giant Ferris wheel at Pier 57 on Elliott Bay. Their vertical height H(t) (in meters) above the surface of Elliott Bay as a function of time t (in seconds) can be modeled with a sinusoidal expression of the form: a·cos(bt) + d At t = 0, when the Wheel starts moving, they are at a height of 5 m above the water surface of Elliot Bay, which is the lowest point of the ride. After 120 seconds, they reach the maximum height of 59 m. Which of the following is the correct function H(t)? A) $H(t) = 27 \cdot cos\left(\frac{\pi}{60}t\right) - 5$ B) $H(t) = 5 \cdot cos\left(\frac{\pi}{120}t\right) + 27$ C) $H(t) = -27 \cdot cos\left(\frac{\pi}{60}t\right) + 5$ D) $H(t) = -27 \cdot cos\left(\frac{\pi}{120}t\right) + 32$ E) Answer not given. **Because they are at the minimum height at t = 0, the function must start with a negative sign. The amplitude of the function is the distance they move up and down from the midline. The distance they move vertically is 59 – 5 = 54, therefore the amplitude is 54/2 = 27. The midline is at the minimum (5) plus the amplitude, or 5 + 27 = 32. The period is 120 seconds times 2, or 240 seconds, the time for a full cycle.**
9		C	What is the sum of all solutions for 'x' in the following equation? $log_4(x+1) + log_4(x+13) = 3$ A) -17 B) -14 C) 3 D) 14 E) Answer not given. $\boldsymbol{log_4(x+1)(x+13) = 3}$ **$(x+1)(x+13)=4^3 = 64$** **$x^2 + 14x - 51 = 0$** **$(x - 3)(x + 17) = 0$** **$x = 3, -17$** **However, x = -17 is not in the domain of the original equation, therefore the only solution is x = 3.**

	9	*A*	What is the domain of the following function? $f(x) = log_{27}\left(log_9\left(log_3\left(log_{1/3}\, x\right)\right)\right)$ A) $\left(0, \frac{1}{27}\right)$ B) $\left(0, \frac{1}{3}\right)$ C) $(0, 1)$ D) $(0, \infty)$ E) Answer not given. **Define the following:** **a = $\log_{1/3}$x** **b = $\log_3$a** **c = $\log_9$b** **f(x) = $\log_{27}$c** **Therefore, c > 0, and $\log_9$b > 0.** **Since the base (9) is > 1, then b > 1, so $\log_3$a > 1. $\log_3$a goes through point (3, 1), therefore a > 3.** **Finally, $\log_{1/3}$x > 3. If you write an equation: $\log_{1/3}$x = 3, then rewrite in exponential form, x = $(1/3)^3$, solve for x = 1/27. Because the logarithmic inequality has a fractional base, the solution that we want is for x < 1/27, and it also has to be > 0.**
10	**10**	*A*	You have two beakers in chemistry lab, one that measures 75 milliliters and one that measures 90 milliliters. You can perform any of the following actions: • Completely fill a container from the tap. • Completely empty a container down the drain. • Pour a container into the other until it is completely full or the one pouring is completely empty. Your goal is to measure out exactly 32 milliliters of water. To accomplish this, you must start with some water in one of the containers. Which of the following amounts must you choose to start with? A) 2 ml B) 4 ml C) 6 ml D) 8 ml E) Answer not given. **Start with 2 ml in the 75 ml container. Fill the 90 ml container, and pour as much as possible (73 ml) into the 75 ml container. There is 17 ml left in the 90 ml container. Empty the 75 ml container, and pour the 17 ml into it. Fill the 90 ml container again. Pour as much as possible (58 ml) into the 75 ml container. There is now 32 ml left in the 90 ml container.**

"Math is Cool" Championships -- 2020-21

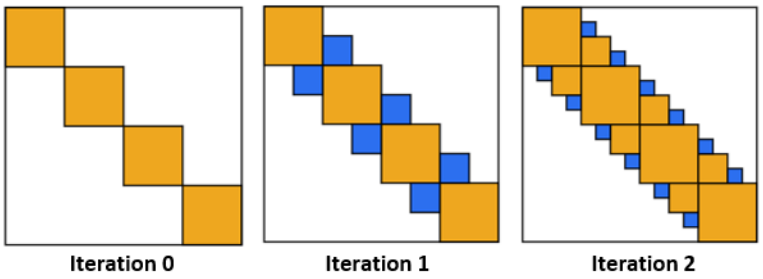
High School

Team Test Solutions

9/ 10th	11/ 12th	Answer	Solution
1	1	2025	If x = 5 and y = 3, calculate the value of: $(9x^4y^6)^{1/2}$ **$3(5^2)(3^3) = 2025$**
2	2	5 [gallons]	Nihal's job is to order food for a camp that has 250 campers attending the camp each week. Each camper eats 5/16 of a cup of gruel each week. Gruel comes in one gallon packages. What is the minimum number of one gallon packages Nihal will need to order for the 1st week of camp? **16 cups in a gallon** **$250/(^{16}/_{16/5}) = 4.88$ gallons**
3		67	Rosa made a geometric solid called a tetrahedron by putting together 4 equilateral triangles with side lengths of 8 units. Find the surface area of the tetrahedron in the form $a\sqrt{b}$, where $\sqrt{b}$ is a simplified radical. What is the value of a + b? **Surface area of a regular tetrahedron =** $\sqrt{3} \cdot a^2 = \sqrt{3} \cdot 8^2 = 64\sqrt{3}$
	3	133	Rosa made a geometric solid called a tetrahedron by putting together 4 equilateral triangles with side lengths of 8 units. Find the volume of the tetrahedron in the form $\frac{a\sqrt{b}}{c}$, where $\sqrt{b}$ is a simplified radical and $\frac{a}{c}$ is a simplified fraction. What is the value of a + b + c? **Volume of a regular tetrahedron =** $\frac{a^3}{6\sqrt{2}} = \frac{8^3}{6\sqrt{2}} = \frac{128\sqrt{2}}{3}$

4	**4**	**36**	One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd size had gone to the mountain slopes, and the remaining 15 camels from the herd stayed on the riverbank. How many camels are in the herd? **The number of camels must be a square number, and also must be divisible by 4. Use trial and error to discover that it must be 36.**
5	**5**	**6**	Hasan and Carli are playing the card game Slap Jack. They are playing a best-of-5 series, where they keep playing games until one person has won 3 games. There are no tie games. If the series lasts 5 games with Hasan winning, in how many different ways could the outcomes of the games occurred? **Hasan won the 5th game, to win the series. Therefore, in games 1-4, Hasan won 2 and Carli won 2. The number of ways is: 4!/(2!2!) = 6.**
6	**6**	**1**	What digit is in the hundreds place of 5^{2021}? **The first few powers are 5, 25, 125, 625, 3125. The hundreds place alternates between 1 (odd powers) and 6 (even).**
7	**7**	**24**	What is the value of the following expression to the nearest integer? $\log_2(9) \cdot \log_3(125) \cdot \log_5(16)$ **Use the change of base formula to rewrite in log base 10:** $= \frac{\log 9}{\log 2} \cdot \frac{\log 125}{\log 3} \cdot \frac{\log 16}{\log 5}$ $= \frac{2\log 3}{\log 2} \cdot \frac{3\log 5}{\log 3} \cdot \frac{4\log 2}{\log 5}$ **= (2)(3)(4) = 24**

8		**5**	A certain number of positive three-digit numbers can be formed from the following set of digits (repetitions are allowed): {0, 1, 2, 3, 4, 5}. If a single three-digit number is chosen at random from all of the three-digit numbers that can be formed, the probability that it is not a multiple of 5 can be written as a reduced fraction A/B. What is A + B ? **There are 180 total numbers that can be formed: 5·6·6** **If the number ends in 0 or 5, it is divisible by 5: 5·6·2 = 60 of them. Therefore, there are 120 numbers that are NOT divisible by 5. P = 120/180 = 2/3**
	8	**5**	If three fair, six-sided dice are rolled, and the sum of the numbers rolled is even, the probability that all three numbers rolled were even can be written as a reduced fraction A/B. What is A + B? **Want P(all even | sum is even)** **There are 3x3x3=27 ways to get all evens. To have an even sum, you need to have EEE, OOE, OEO, or EOO. There are also 27 ways to get each of these. Therefore the probability is 27/(4*27) = 1/4.**

9	**9**	**23**	A fractal is created by starting with 4 identical squares, arranged along the diagonal of a larger square, as shown in Iteration 0. In Iteration 1, the corners between adjacent squares are filled with squares that have half the side length of the original four squares. The same method is used to create Iteration 2. If the pattern continues, the fraction of the larger square that is shaded after infinitely many iterations can be written as a reduced fraction A/B. What is A + B ?  Iteration 0 Iteration 1 Iteration 2 **Assume the original square has sides of length 1 unit. At Iteration 0, the shaded area = $\frac{1}{4}$. At Iteration 1, the additional shaded area = 3/32. At Iteration 2, the additional shaded area is (3/32)*(1/2). If you investigate one more iteration, you will see that the additional shaded area is (3/32)*(1/2)*(1/2). Therefore, from Iteration 1 on can be represented with an infinite geometric series with r = $\frac{1}{2}$. The sum of the infinite series = $\frac{\frac{3}{32}}{1-1/2} = \frac{3}{16}$** **Add this to the original shaded area in Iteration 0: $\frac{1}{4}$ + 3/16 = 7/16.**
10		**2**	Suppose g(x) is a quadratic polynomial $ax^2 + bx + c$, such that g(3) = 14, g(4) = 23, and g(5) = 34. What is a + b + c ? **Plug the x values 3, 4, and 5 into the quadratic: $g(x) = ax^2 + bx + c$** **9a + 3b + c = 14** **16a + 4b + c = 23** **25a + 5b + c = 34** **Solve the equations for:** **a = 1, b = 2, c = -1**

	10	**8**	What is the minimum value of the function: $$f(x) = \left\lvert \frac{tan^2x + 1}{sin^3x \cos x} \right\rvert$$ $$\boldsymbol{f(x) = \frac{tan^2x + 1}{sin^3x \cos x}}$$ $$\boldsymbol{= \frac{sec^2(x)}{sin^3x \cos x} = \frac{1}{sin^3x\, cos^3\, x} = \frac{1}{(sinx\, cosx)^3}}$$ $$\boldsymbol{= \frac{1}{\left(\frac{\sin 2x}{2}\right)^3}}$$ **The minimum positive value will occur when sin 2x is a maximum, or equal to 1. Therefore,** $$\boldsymbol{f(x) = \frac{1}{\left(\frac{1}{2}\right)^3} = 8}$$

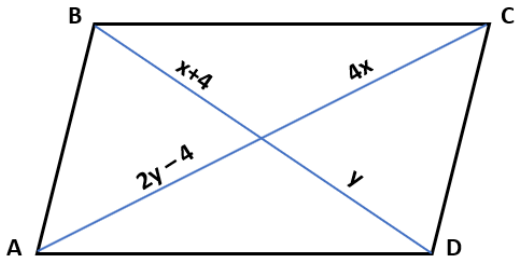
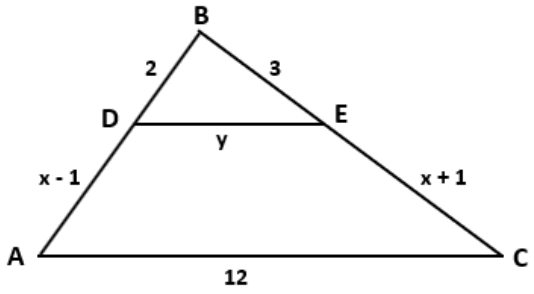
"Math is Cool" Championships -- 2020-21

High School

Pressure Round Solutions

9/ 10th	11/ 12th	Answer	Solution
1	**1**	9 [jelly beans]	In a jar of red, green, and yellow jelly beans, all but 6 are red jelly beans, all but 8 are green jelly beans, and all but 4 are yellow jelly beans. How many jelly beans are in the jar? **Write three equations:** **G + Y = 6** **R + Y = 8** **G + R = 4** **Solve for G = 1, R = 3, Y = 5**
2	**2**	264 [seats]	Row A in a certain section in a baseball stadium has 12 seats. Each row behind row A is labeled alphabetically (B, C, D, and so on) and has 6 more seats than the row immediately in front of it. How many total seats are in rows A through H of this section of the stadium? **Arithmetic series: $a_1 = 12$, d = 6** **$S_8 = \frac{8}{2}(2 \cdot 12 + (8-1) \cdot 6)$** **$S_8 = 264$**
3		[A + B =] 5	Genevieve and Raj are playing a game by flipping a fair coin. The first one to flip the coin and have it show heads wins. Genevieve goes first and they take turns until someone wins. The probability that Genevieve wins the game can be written as a reduced fraction A/B. What is A + B ? **In the first round, Genevieve has a 1/2 chance of winning right away. If she does not then Raj has a 1/2 chance so his probability of winning in the first round is (1/2)(1/2)=1/4. One could set up an infinite series to include the chance of Genevieve winning in all rounds but each round is the same conditioned on making to that round. Looking at the first round, Genevieve has twice the chance of winning as Raj. Her probability will then be 2/3 and Raj 1/3. 2+3=5. Or, the infinite series for Genevieve winning is: ½ + 1/8 + 1/32 + … The sum of the infinite series is 2/3.**

	3	[A =] 10	The product symbol $\prod$ is used to express the product of a sequence of numbers. For example: $\prod_{k=1}^{2}(k+1) = (1+1)(2+1) = 6$ The following product can be written as A.B, where A is an integer (number of digits not specified), and B is the decimal portion of the number. Find A. $\prod_{k=2}^{2021} log_k(k+1)$ $\prod_{k=2}^{2021} \boldsymbol{log_k(k+1)} =$ **$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_{2021} 2022$** $= \log_2 3 \cdot \frac{\log_2 4}{\log_2 3} \cdot \frac{\log_2 5}{\log_2 4} \cdot \ldots \cdot \frac{\log_2 2022}{\log_2 2016}$ **$= \log_2 2022 = x$** **$2022 = 2^x$** **$2^{10} = 1024$** **$2^{11} = 2048$** **Therefore, x = 10.something, so the integer portion 'A' of the number is 10.**
4	**4**	[x =] 8	The surface of a cube of integer edge length 'x' is painted red and then cut into x3 identical smaller cubes. If the number of small cubes that have no side painted and the number of small cubes that have one side painted are the same, what is the value of x? **It is a cube with side lengths 8, which can be discovered with some guess and check for one method. The 8x8x8 cube will have 216 interior cubes that are not painted, and will have 216 exterior cubes that have one side painted.**

5		16	In the following parallelogram, find the length of diagonal AC.  **Each half of a diagonal is equal:** **x + 4 = y** ➔ **x – y = -4** **2y – 4 = 4x** ➔ **4x – 2y = -4** **Solve the system of equations for:** **x = 2, y = 6** **AC = 2y – 4 + 4x = 16**
	5	20	In the following triangle, segment DE is parallel to segment AC. Find the product of x and y.  **Use similar triangles to solve for x = 5 and y = 4.**

"Math is Cool" Championships -- 2020-21

High School

College Bowl Round #1 Solutions

	Answer	Solution
1	12	How many positive factors does 132 have? **$132 = 2^2 3^1 11^1$** **(3)(2)(2) = 12**
2	60	What is the area of a rhombus with vertices at (8, 0), (14, -5), (8, -10) and (2, -5)? **Area = pq/2 = (10)(12)/2 = 60**
3	888	Find the sum of the first twelve terms of the following arithmetic sequence: 8, 20, 32, … **Partial sum of an arithmetic series: Sn = n/2(2a1 + (n-1)d)** **= 12/2(2·8 + 11·12) = 888**
4	25	Every day, Fatima saves a penny, a dime and a quarter. What is the least number of days required for her to save an amount equal to an integral (counting) number of dollars? **Penny + dime + quarter = 36 cents. Need to get to some multiple of 100 cents. LCM of 36 and 100 = 900. It will take 900/36 = 25 days.**
5	3	A game is played as follows. A single card is drawn from a standard 52-card deck. If the card is red or a King, the player wins $21. If the card is NOT red or a King, the player loses $18. What is the expected value of the game, in dollars? **P(red or K) = 28/52** **P(NOT red or K) = 24/52** **E(x) = (28/52)(21)+(24/52)(-18) = 3**
6	968	Consider the set of numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. How many subsets of this set have more than two elements? **Total number of subsets = 210 = 1024.** **0 element sets: 1 (empty set)** **1 element sets: 10** **2 element sets: 10C2 = 45** **1024 – (1 + 10 + 45) = 968**

7	5	If the sum of two numbers is thirty, and the product is six, what is the sum of the reciprocals of the two numbers? **1/x + 1/y = (x+y)/(xy) = 30/6 = 5**
8	10	Six identical cookies are divided among three children. If each child gets at least one cookie, in how many ways can the cookies be distributed? **Three cookies are initially distributed, leaving three. Use the stars and bars method: 3 stars (cookies) + 2 bars (dividers) = 5. 5C2 = 10**
9	182	An ant farm starts with 50 ants. It doubles in size the first month, increases by 30 ants in the second month, and increases by 40% in the third month. How many ants are there after the third month? **50x2 = 100** **100+30 = 130** **130(1.4) = 182**
10	16	What is the sum of the digits of 11 raised to the fourth power? **114 = 14641, row 4 from Pascal's triangle.**

"Math is Cool" Championships -- 2020-21

High School

College Bowl Round #2 Solutions

	Answer	Solution
1	32	A square is inscribed in a circle of radius 4. What is the area of the square? **Diameter of circle = diagonal of square = 8. Area = (8)(8)/2 = 32**
2	22	How many integers are between the cube root of 25 and the square root of 600? **First integer will be cube root of 27 = 3, last integer will be 24, because 600 is between 242 and 252. Therefore, number of integers between = 24 – 2 = 22**
3	100	A solid box is 8 inches by 11 inches by 12 inches and is made of 1-inch cubes. If the outside of the box is painted, how many unit cubes have exactly two of their faces painted? **Edges that are 8 inches (4 of them) have 6 cubes with 2 painted sides. Edges that are 11 inches (4 of them) have 9 cubes with 2 painted sides. Edges that are 12 inches (4 of them) have 10 cubes with 2 painted sides. Therefore, the total is (6+9+10)·4 = 100**
4	41	A father has two children whose ages differ by 5 years. The sum of the squares of their ages is 97. The square of the father's age can be found by writing the squares of the children's ages one after the other as a four-digit number. How old is the father? **The children are 4 and 9, 16 + 81 = 97. Write: 1681, the square root is 41.**
5	5040	How many ways are there for a teacher to hand out one marker each to four different students, out of a set of 10 different colored markers? **10P4 = 5040**

6	10	Three of the vertices of a cube are connected to form a triangle. The probability that all three vertices are on the same face of the cube can be written as a reduced fraction A/B. What is A + B? **The total number of triangles possible is 8C3 = 56. On one face, the total number of triangles possible is 4C3 = 4. Multiply that by 6 faces to get 24 possible triangles. P = 24/56 = 3/7. A + B = 10.**
7	2	The radius of a cylinder is increased by 40%, and the height of the cylinder is cut in half. What is the absolute value of the percent change in the volume of the cylinder? **Increasing the radius by 40% increases the area of the base by (1.4)2 or 1.96 times the original. Multiplying by ½ the height gives 0.98 times the original volume, which is a 2% decrease. The absolute value of the percent change is 2.**
8	0	What is the 2021st digit in the decimal expansion of 1/41? **1/41 = $0.\overline{02439}$** **The five digits repeat in an infinite series. Therefore, the 2020th digit will be 9, and the 2021st digit will be 0.**
9	52	A landscape maintenance service is spraying for weed control. Eight ounces of the chemical being used treats 500 square feet of lawn. How many ounces of the chemical will be needed for treating 3250 square feet of lawn? **8/500 = x/3250** **x = 52**
10	4095	The Mersenne numbers are generated by the formula $M_n = 2^n - 1$, for n = 1, 2, 3, … What is the 12th Mersenne number? **$M_{12} = 2^{12} - 1 = 4096 - 1 = 4095$**

"Math is Cool" Championships -- 2020-21

High School

College Bowl Round #3 Solutions

	Answer	Solution
1	224	What is the sum of the positive factors of 84? **$184 = 2^2 3^1 7^1$** **Sum of factors = $(2^0+2^1+2^2)(3^0+3^1)(7^0+7^1)$**
2	96	A glass is three-fourths full of water, and one-eighth of the glass capacity is emptied to completely fill a 16 ounce container. How much water was originally in the first glass? **If 1/8 capacity = 16 ounces, then 16*8 = 128 ounces is the full capacity. 128*(3/4) = 96.**
3	1806	The mathematician Augustus De Morgan lived in the nineteenth century. It is claimed that he made the following statement: " I was 'x' years old in the year x^2 ". In what year was the mathematician born? **The nineteenth century is the 1800's. By trial and error, his age must have been 43, since $43^2 = 1849$. So if he was 43 in 1849, then the year of his birth was 1849 – 43 = 1806.**
4	12	Shonda can bike from home to work in 1.2 hours. By bus, the trip takes 0.4 hours. If the bus travels 20 miles per hour faster than Shonda rides her bike, how many miles is it from Shonda's home to work? **Let x = rate on the bike** **1.2x = 0.4(x+20)** **Solve for x = 10 mph** **Distance = 1.2(10) = 12 miles**

5	1657	The weatherman has predicted a 30% chance of snow for each of the next three days. The probability that it will snow on at least one of the next three days can be written as a reduced fraction A/B. What is A + B? **P(at least one) = 1 – P(none)** **P(no snow) = $(7/10)^3$ = 343/1000** **1 – 343/1000 = 657/1000** **A + B = 1657**
6	56	There are 15 points on a chalkboard numbered 1 through 15. How many lines must be drawn to connect every pair of points whose sum is odd, assuming each line connects only one pair of points? **Need one odd and one even number to get an odd sum. Each odd number (8 of them) must be connected to each even integer (7 of them). (8)(7) = 56.**
7	19	An isosceles trapezoid has base angles of 60 degrees. The lengths of the bases are 5 and 21 units. What is the length of one of its diagonals? **Drop a perpendicular from one of the top vertices to form a 30-60-90 triangle. Solve for the height, which equals $8\sqrt{3}$. From the same top vertex, draw the diagonal to the opposite vertex on the bottom, which forms a right triangle with height $8\sqrt{3}$ and length 13. Use Pythagorean theorem to solve for the hypotenuse (diagonal) = 19.**
8	16	When six gallons of gasoline are put into a car's tank, the indicator goes from $\frac{1}{4}$ of a tank to 5/8 of a tank. What is the total capacity of the gasoline tank in gallons? **(1/4)x + 6 = (5/8)x** **x = 16**
9	2	What is the positive difference between the median and the mean of the following numbers: 19, 25, 20, 8, and 13? **Mean = 17, median = 19**
10	15	What is the length of the space diagonal of a rectangular prism with sides of length 14, 5 and 2? **$14^2 + 5^2 + 2^2 = 225$** **Square root of 225 = 15**